臺灣大學數學系
105 學年度上學期博士班資格考試題
科目：偏微分方程

2016.09.23

1. (25 pts)
(a) Solve

$$\begin{cases} u_x + u_y = \dfrac{1}{u}, & x \in \mathbb{R}, y > 0, \\ u(x,0) = \dfrac{1}{1+x^2}, & x \in \mathbb{R}. \end{cases}$$

(b) Find the weak solution $u(x,t)$ of the equation

$$\begin{cases} u_t + (u^2)_x = 0 \; x \in \mathbb{R}, t > 0, \\ u(x,0) = g(x), \; x \in \mathbb{R} \end{cases}$$

which satisfies the entropy condition, where

$$g(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 2 & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x > 1. \end{cases}$$

(Assume the correponding conservation law of the equation is

$$0 = \frac{d}{dt} \int_a^b u(x,t)\, dx + u^2(b,t) - u^2(a,t)$$

for any $a, b \in \mathbb{R}$.)

2. (25 pts)
(a) Prove the Poincare inequality on $[0,1]$: Assume that u is a C^1 function with $u(0) = u(1) = 0$. Then there exists a constant C independent of u such that

$$\int_0^1 |u(x)|^2\, dx \leq C \int_0^1 |\nabla u(x)|^2\, dx.$$

(b) Let u satisfy

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), \; x \in [0,1], t > 0, \\ u(0,t) = u(1,t) = 0, \; t > 0, \\ u(x,0) = x(1-x), \; x \in [0,1]. \end{cases}$$

Show that (1) $\int_0^1 u^2(x,t)\, dx$ tends to 0 exponetially as $t \to \infty$; (2) $u(x,t)$ tends to 0 as $t \to \infty$ for each $x \in [0,1]$.

3. (25 pts) Let u be a C^2 solution of $\Delta u = 0$ in a domain $\Omega \subset \mathbb{R}^2$.
(a) Prove that $\max_{\Omega} u \leq \max_{\partial\Omega} u$ if Ω is bounded.
(b) Let $\Omega = \{(x,y) \,|\, x^2 + y^2 > 1\}$. Show that the solution u is unique if $u(x,y) = 0$ on $|(x,y)| = 1$ and $\lim_{|(x,y)| \to \infty} u(x,y) = 5$.
(c) Let $\Omega = \{(x,y) \,|\, -\infty < x < \infty, 0 < y < 1\}$. Show that the solution u is a constant if $\frac{\partial}{\partial y} u(x,y) = 0$ for $y = 0$ or 1, and $u(x,y) > 0$ in Ω.

4. (25 pts) Solve the equation

$$\begin{cases} u_{tt}(x,t) - 3u_{xt}(x,t) + 2u_{xx}(x,t) = 0 \; \text{ in } \mathbb{R}^2, \\ u(x,0) = x, u_t(x,0) = 1 - x \; \text{ for } x \in \mathbb{R}. \end{cases}$$