臺灣大學數學系
104 學年度上學期博士班資格考試題
科目：偏微分方程 2015.10.02

PDE Qualifying Exam Fall 2015

1. (20 points) Let u be a smooth function that solves the wave equation

$$u_{tt} - \triangle u = 0 \qquad \text{in } \mathbb{R}^n \times (0, \infty),$$

where the integer $n \geq 2$. Let $x \in \mathbb{R}^n$, $t > 0$, $r > 0$. Define

$$U(x; r, t) := ⨍_{\partial B(x,r)} u(y, t) dS(y),$$

the average of $u(\cdot, t)$ over the sphere $\partial B(x, r)$. Show that for fixed $x \in \mathbb{R}$, U satisfies

$$U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 \qquad \text{in } \mathbb{R}_+ \times (0, \infty).$$

2. (20 points) State and prove the Lax-Milgram Theorem.
3. (20 points) Let $U \subset \mathbb{R}^N$ be a smooth and bounded open set and suppose $f \in L^2(U)$, where N is a positive integer. Show that there exists a unique weak solution $u \in H_0^1(U)$ of the following boundary-value problem

$$\begin{cases} \triangle u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

4. (20 points) Do you think the following second order differential equations have periodic solutions ? Prove or disprove it.

$$x''(t) + x(t) - x^2(t) = 0, \tag{0.1}$$

$$x''(t) + x^3(t) + x^4(t) = 0. \tag{0.2}$$

5. (20 points) Solve the following partial differential equation

$$\begin{cases} u u_{x_1} + u_{x_2} = 1, & \text{in } \mathbb{R}^2, \\ u(x_1, x_1) = \frac{1}{2} x_1. \end{cases}$$