博士班資格考 (PDE) Fall 2013

1. (20%) Solve using characteristics:

   (i) $u_t + u_x = u^2$, for $x \in \mathbb{R}, t > 0$, and $u(x,0) = g(x)$ for $x \in \mathbb{R}$

   (ii) $xu_t - tu_x = u$, for $x \in \mathbb{R}, t > 0$, and $u(x,0) = g(x)$ for $x \in \mathbb{R}$,
   where g is a smooth function with compact support.

2. (20%) Given a continuous function, $f \in C^0(\bar{\Omega})$ where Ω is a bounded smooth domain in $\mathbb{R}^n, n \geq 2.$ Consider the poisson equation

$$\Delta u = f \quad \text{in } \Omega.$$

Must the corresponding solution $u \in C^2(\bar{\Omega})$? What about $u \in C^2_{loc}(\Omega)$? Prove or disprove all your answers.

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3. (20%) Consider the following problem:

$$(1)\quad \begin{cases} -\Delta u_i + \sum_{j=1}^{N} a_{ij} u_j = f_i \ \text{ in } \Omega, \\ u_i = 0 \ \text{ on } \partial\Omega, \ i = 1, \cdots, N \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n, n \geq 2$,

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ is a symmetric and positive definite matrix, and each $f_i \in L^2(\Omega)$ is a nonzero function.

Prove that there exists a unique solution of problem (1).

4. (20%) Consider the following problem:

$$(2)\quad \begin{cases} \frac{\partial u_i}{\partial t} - \Delta u_i + \sum_{j=1}^{N} a_{ij} u_j = 0 \ \text{ for } x \in \Omega,\ 0<t<T, \\ u_i = 0 \ \text{ for } x \in \partial\Omega, 0 < t < T, \ i = 1, \cdots, N, \\ u_i = g_i \ \text{ for } x \in \Omega, t = 0, i = 1, \cdots, N, \end{cases}$$

where $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ is a symmetric real-valued matrix and each $g_i \in L^2(\Omega)$ is a nonzero function.

Prove that there exists a unique solution of problem (2).

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5. (20%) Let u solve

$$\begin{cases} u_{tt} = \Delta u \text{ in } \mathbb{R}^3 \times (0,\infty), \\ u_t = h \text{ on } \mathbb{R}^3 \times \{t=0\}, \\ u = g \text{ on } \mathbb{R}^3 \times \{t=0\}, \end{cases}$$

where g, h are smooth and have compact support.

Prove that

$$\sup_{x\in\mathbb{R}^3, t>0} |tu(x,t)| < \infty$$

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Prove that

$$\sup_{x\in\mathbb{R}^3, t>0} \left| t u(x,t) \right| < \infty$$