臺灣大學數學系

101 學年度上學期博士班資格考試題

科目：偏微分方程　　2012.09.14

Choose 4 from the following 5 problems

1. (25 pts) Consider the following conservation law

$$\begin{cases} u_t(x,t) + [F(u(x,t))]_x = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

   (a) Define weak solutions for the equation.

   (b) Derive the Rankine-Hugoniot condition for u along a discontinuity curve.

2. (25 pts) Let Ω be an open set in $\mathbb{R}^n$. Assume $v \in C^2(\bar{\Omega})$ is subharmonic, that is, $\Delta v \geq 0$ in Ω.

   (a) Show that

$$v(x) \leq \frac{1}{|B(x,r)|}\int_{B(x,r)} v(y)\,dy \quad \text{for all } B(x,r) \subset \Omega,$$

   where $|B(x,r)|$ denotes the volume of $B(x,r)$.

   (b) Prove that $\max_{\bar{\Omega}} v = \max_{\partial\Omega} v$.

   (c) Prove that $|\nabla u|^2$ is subharmonic if u is harmonic.

3. (25 pts) Assume for $(x,t) \in \mathbb{R}\times(0,\infty)$

$$\begin{cases} 2tu_t(x,t) + xu_x(x,t) = 0, \\ u_t(x,t) = u_{xx}(x,t). \end{cases}$$

   Solve u.

4. (25 pts) Solve

$$\begin{cases} 2u_{tt}(x,t) - u_{xt}(x,t) - u_{xx}(x,t) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = 4x^2 + x,\ u_t(x,0) = 1 - 4x & \text{for } x \in \mathbb{R}. \end{cases}$$

5. (25 pts) Let $A = [0,a]\times[0,b] \subset \mathbb{R}^2$ and u be a C^1 function on A with $u = 0$ on ∂A.

   (a) Prove the Poincare inequality: there exists a constant C independent of u such that

$$\int_A |u(x)|^2\,dx \leq C\int_A |\nabla u(x)|^2\,dx.$$

   (b) What is the best constant C?