台灣大學數學系
九十九學年度下學期博士班資格考試題
科目：數值偏微分方程
2月25日，2011

Total score: 100 points

1. (20 points) Consider the initial value problem for an ordinary differential equation of the form:

$$\frac{du(t)}{dt} = f(u(t),t), \qquad u(0) = u_0,$$

where $t \in [0,T]$ is the time variable, $T > 0$, $u \in \mathbb{R}^m$ is a real-valued vector function, and the function $f \in \mathbb{R}^m$ is assumed to be continuous with respect to t and uniformly Lipschitz continuous with respect to u for $t \in [0,T]$, yielding the existence and uniqueness of the solution for this problem.

Denote U^n and F^n to be the numerical approximation of u and f at time $t_n = nk$, respectively, where k is the time step. Which of the following linear multistep methods are convergent when each of them are used to approximate this problem numerically ? For the ones that are not, are they inconsistent, or not zero-stable, or both?

(a) $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kF^{n+1}$,

(b) $U^{n+1} = -4U^n + 5U^{n-1} + k\left(4F^n + 2F^{n-1}\right)$,

(c) $U^{n+4} = U^n + \frac{4}{3}k\left(F^{n+3} + F^{n+2} + F^{n+1}\right)$,

(d) $U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k\left(F^{n+2} + F^{n+1}\right)$.

2. (25 points) Consider the following finite-difference method for solving the heat equation $\partial_t u(x,t) = \partial_{xx}u(x,t)$ on a unit domain $x \in [0,1]$,

$$U_j^{n+2} = U_j^n + \frac{2k}{h^2}(U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}),$$

with the initial and periodic boundary conditions $u(x,0) = u_0(x)$ and $u(0,t) = u(1,t)$, respectively. Here U_j^n denotes the approximate solution of u at a spatial point $x_j = jh$ and at a time $t_n = nk$, where h and k are the mesh sizes in the spatial and temporal domains, respectively.

(a) Determine the order of accuracy of this method (in both space and time).

(b) Suppose we take $k = \alpha h^2$ for some fixed $\alpha > 0$ and refine the grid. For what values of α (if any) will this method be Lax-Richtmyer stable and hence convergent?

(c) Is this a useful method ?

3. (30 points) Consider the Poisson problem $\nabla^2 u = f$ on a unit square domain $D \in [0,1] \times [0,1]$ with the Dirichlet boundary condition $u|_{\partial D} = g$. Devise a higher-order method for this problem (that is, the order of accuracy of the proposed method is higher than 2); algorithmic details are required.

4. (25 points) The Camassa-Holm equation,

$$\partial_t u + 2\kappa \partial_x u - \partial_{xxt} u + 3u\partial_x u - 2\partial_x u \partial_{xx} u - u\partial_{xxx} u = 0,$$

results from an asymptotic expansion of the Euler equations governing the motion of an inviscid fluid whose free surface can exhibit gravity-driven wave motion, where $\kappa \in \mathbb{R}$ is a constant. Suppose that we are interested in an initial value problem with $u(x, 0) = u_0$ as the initial condition for $x \in [0, 1]$, and the periodic boundary conditions on both sides.

To find approximate solution of this problem numerically, one approach is to first rewrite the above Camassa-Holm equation into the form

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2\right) + \partial_x p = 0$$

with

$$p - \partial_{xx} p = \kappa u + u^2 + \frac{1}{2} (\partial_x u)^2 .$$

Note that if $\partial_x p = 0$, we have the inviscid Burgers equation, while if u is known a priori we have the Helmholtz equation for p. Now based on the above couple systems, devise a "convergent" numerical method for this problem.