台灣大學數學系
九十七學年度第下學期博士班資格考試題
科目：數值偏微分方程
February 26, 2009

Total score: 100 points

1. (25 points) Consider Poisson's equation

$$\begin{cases} -\triangle u = f(x,y),\ (x,y) \in D = (0,1) \times (0,1) \\ u = 0 \text{ on } \partial D. \end{cases} \tag{1}$$

   (a) (10 points) Derive a second order finite difference method to solve the equation.

   (b) (5 points) Show that the corresponding matrix is positive definite.

   (c) (10 points) Outline an iterative method that is suitable to solve the resulting linear system. Justify why the iterative method is chosen.

2. Consider the problem

$$-u''(x) = f(x) \tag{2}$$

where $x \in (0,1)$, $u(0) = u(1) = 0$, and $f(x) \in H_0^1$. Consider the following hat functions ϕ_j's for $j = 1, ..., M-2$, $h = \frac{1}{M-1}$, and $x_j = jh$ that form the basis of a subspace $S_h \subset H_0^1$.

$$\phi_j = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}},\ x = [x_{j-1}, x_j], \\ \frac{x - x_{j+1}}{x_j - x_{j+1}},\ x = [x_j, x_{j+1}], \\ 0, \text{otherwise.} \end{cases} \tag{3}$$

Define the approximate solution in S_h:

$$u_h = \sum_{j=1}^{M-2} u_j \phi_j(x). \tag{4}$$

   (a) (15 points) Let $M = 6$. Derive a Galerkin approach finite element method in detail to solve the problem by writing down the resulting linear system.

   (b) (10 points) Write a pseudo-code to generate the linear system for general M. You may use your result in (a).

3. (50 points) Consider the one-dimensional wave equation of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 \in \mathbb{R} > 0, \tag{5}$$

with the initial conditions

$$u(x,0) = u_0(x),$$
$$\frac{\partial u}{\partial t}(x,0) = u_1(x),$$

and a periodicity condition

$$u(x,t) = u(x+L,t), \qquad L \in \mathbb{R},$$

as boundary conditions when $x \in [0, L]$.

(a) (20 points) One possible way to solve this problem numerically is to discretize (5) directly using a finite difference formula as

$$U_j^{n+1} - 2U_j^n + U_j^{n-1} = \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right). \tag{6}$$

Here $U_j^n \approx u(x_j, t_n)$ is the approximate solution of u at point x_j and at time t_n, and Δx, Δt are the spatial and temporal mesh sizes, for x and t, respectively. Examine the consistency and stability conditions of the scheme (6).

(b) (30 points) An alternate way to solve this problem numerically is to rewrite (5) as a first order system, and devise a numerical scheme for that. Now give a complete description on how this can be done for the development of a stable and second-order accurate scheme.