臺灣大學數學系

九十七學年度上學期博士班資格考試題

科目：數值偏微分方程 2008.09.18

Total score: 100 points

1. (50 points) Consider the method

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1}) \tag{1}$$

for the advection equation $u_t + au_x = 0$ on $0 \le x \le 1$ with periodic boundary conditions. Here U_j^n denotes the numerical approximation of the exact solution $u(x_j, t_n)$ at the point x_j and time t_n, and h, k are the spatial and temporal step sizes, respectively.

(a) This method can be viewed as the trapezoidal method applied to an ODE system $U'(t) = AU(t)$ arising from a method of lines discretization of the advection equation. What is the matrix A?

(b) Suppose we want to fix the Courant number $\nu = ak/h$ as $k, h \to 0$. For what range of Courant numbers will the method be stable if $a > 0$? If $a < 0$? Justify your answers in terms of eigenvalues of the matrix A from part (a) and the stability regions of the trapezoidal method.

(c) Apply von Neumann stability analysis to the method (1). What is the amplification factor ?

(d) For what range of ν will the CFL condition be satisfied for this method (with periodic boundary conditions)?

(e) Derive the modified equation for the method (1) and show that it is second order accurate.

(f) Suppose we use the same method (1) for the initial-boundary value problem with $u(0,t) = g_0(t)$ specified. Since the method has a one-sided stencil, no numerical boundary condition is needed at the right boundary. For what range of ν will the CFL condition be satisfied in this case? What are the eigenvalues of the A matrix for this case and when will the method be stable?

2. (50 points) Consider the Poisson equation in polar coordinates of the form

$$\begin{cases} u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta} = f(r,\theta), & 0 < r < 1, \quad 0 \le \theta \le 2\pi, \\ u(\cos\theta, \sin\theta) = g(\theta), & 0 \le \theta \le 2\pi, \end{cases} \tag{2}$$

where f and g are some specified real-value functions.

(a) To solve this problem numerically, devise a numerical algorithm based on a five-point finite difference scheme with the appropriate boundary conditions. Give comments on the amount of flops that will be required, when the method is carrying out in a computer.

(b) Devise a Fourier based fast Poisson solver to solve this problem. How fast is this method as in comparison with the traditional method in (a) ?