國立臺灣大學數學系

九十六學年度上學期博士班資格考試題

科目：數值 PDE

2007.09

1. (10 points) In your viewpoint, what is the most fundamental theory or the most exciting idea in numerical methods for solving partial differential equations? State the theory or the idea and then explain why you think so.

2. (10 points) Consider the following one dimensional time-dependent wave equation $u_t = -cu_x$, for a nonzero constant c.
(i) Suppose the initial condition $u(0,x)$ is sketched in Figure 1 below for c=1. Carefully illustrate the characteristics of the solutions as time evolves by sketching the solutions.
(ii) Briefly explain how the change in the constant c will affect the solutions.

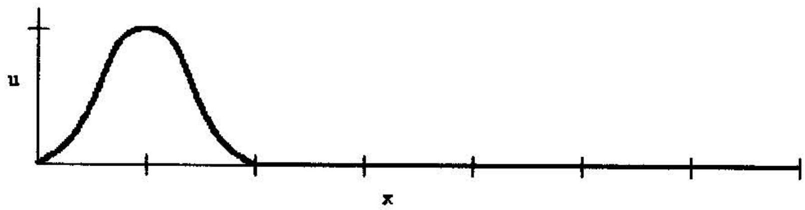

Figure 1.

3. (50 points) Suppose we want to solve the Poisson problem $\nabla^2 u = f$ on the unit square $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ with Dirichlet boundary conditions $u|_{\partial D} = g$ by using a five-point finite difference scheme over a uniform Cartesian grid (x_i, y_i), for $i = 1, \ldots n$.
(i) (12 points) Derive the discretization scheme in matrix form by using the red-black ordering as shown Figure 2.
(ii) (10 points) Discuss the accuracy of the scheme.
(iii) (10 points) Discuss the stability of the scheme.
(iv) (10 points) Write down a pseudo-code to explicitly generate the matrix and right-hand-side in the resulting linear system of the discretization scheme.
(v) (4 points) Discuss the storage requirements of your pseudo-code in part (iv). Give your answer in terms of n.
(vi) (4 points) What is the floating-point operations count of a matrix-vector multiplication that uses the matrix generated in part (iv)? Give your answer in terms of n.

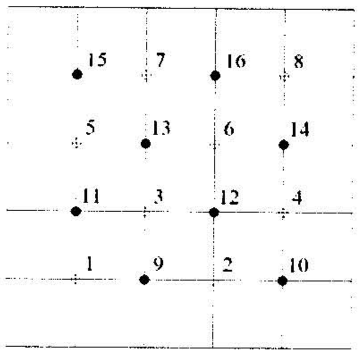

Figure 2.

4. (15 points) Consider a well-defined problem $u''(x) = f(x)$ on $[0,1]$ by assuming $u(0) = u(1) = 0$ and $f(x)$ be a smooth function satisfying $f(0) = f(1) = 0$. Explain how you can solve the problem by a spectral method that approximates the solution by a Fourier series.

5. (15 points) Suppose we use a certain finite difference scheme to solve the PDE $-\Delta u = (2 + \pi^2 y(1-y))\sin \pi x + (2 + \pi^2 x(1-x))\sin \pi y$ over a unit square domain in two space dimensions with the Dirichlet boundary condition. The PDE has the exact solution $u(x,y) = y(1-y)\sin \pi x + x(1-x)\sin \pi y$. After the discretization, we solve the resulting linear system by five different linear system solvers: Cholesky method, Jacobi method, Gauss-Seidel method, SOR method with optimal ω, and Conjugate gradient method. Observe Tables A-E and answer the following questions. Note that in the tables, h stands for the grid size and FLOPs stands for the number of floating-point operations.
(i) (3 points) What is the order of the accuracy of the finite difference scheme?
(ii) (12 points) Match Tables A-E with the five linear solvers and state your reasons.

Table A			
h^{-1}	Iterations	Error	Number of FLOPs
4	6	1.3 E-02	990
8	15	3.3 E-03	13,328
16	33	8.0 E-04	134,100
32	72	2.0 E-04	1,247,378
64	158	4.9 E-05	11,295,774
128	354	1.3 E-05	102,806,246

Table B			
h^{-1}	Iterations	Error	Number of FLOPs
4	15	1.1 E-02	1,908
8	82	2.5 E-03	56,350
16	406	6.3 E-04	1,279,350
32	1,917	1.6 E-04	25,793,240
64	8,827	3.9 E-05	490,489,020
128	39,921	9.8 E-06	9,014,433,584

Table C			
h^{-1}	Iterations	Error	Number of FLOPs
4	N/A	6.8 E-03	322
8	N/A	1.7 E-03	6,974
16	N/A	4.2 E-04	122,230
32	N/A	1.0 E-04	2,029,286
64	N/A	2.6 E-05	33,021,382
128	N/A	6.5 E-06	532,642,694

Table D			
h^{-1}	Iterations	Error	Number of FLOPs
4	3	6.8 E-03	291
8	5	1.7 E-03	3,045
16	9	4.2 E-04	27,465
32	17	1.0 E-04	232,593
64	39	2.6 E-05	2,270,331
128	86	6.5 E-06	20,596,860

Table E			
h^{-1}	Iterations	Error	Number of FLOPs
4	9	1.2 E-02	1,152
8	42	2.6 E-03	28,910
16	204	6.3 E-04	643,050
32	960	1.6 E-04	12,917,762
64	4,415	3.9 E-05	245,331,828
128	19,962	9.8 E-06	4,507,571,630