國立臺灣大學數學系
九十五學年度博士班資格考試試題
科目：數值偏微分方程

2007.06.01

1. (30 points) Consider the initial value problem for the linear advection equation of the form

$$q_t + a\, q_x = 0, \qquad x \in (-1, 1), \quad t > 0, \tag{1}$$

with the initial condition $q(x, 0) = q_0(x)$, and periodic boundaries $q(-1, t) = q(1, t)$; $a \in \mathbb{R}$. Here q_ι denotes the partial derivative of q with respective to ι, for $\iota = x$ and t.

(a) (10 points) Derive the Lax-Wendroff method for numerical approximation of (1).

(b) (15 points) Find the local truncation error as well as the stability condition of the Lax-Wendroff method derived in (a).

(c) (5 points) Discuss the convergence of the method under mesh refinement.

2. (25 points) Suppose that (1) is replaced by the inviscid Burger's equation

$$q_t + (q^2/2)_x = 0, \tag{2}$$

with the same initial and boundary conditions as before.

(a) (5 points) Devise a conservative flux-difference scheme for numerical approximation of (2).

(b) (10 points) Find the local truncation error as well as the stability condition of the numerical method derived in (a).

(c) (10 points) What about the convergence of the method in this case ?

3. (15 points) Consider the finite-difference scheme

$$Q_j^{n+1} = Q_j^n + \frac{\Delta t}{(\Delta x)^2}\left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right) - b\frac{\Delta t}{2\Delta x}\left(Q_{j+1}^n - Q_{j-1}^n\right)$$

for the convection-diffusion equation $q_t = q_{xx} - bq_x$ which is assumed to be well-posed with suitable initial and boundary conditions, $b \in \mathbb{R} > 0$. Here Q_j^n denotes the numerical approximation of the exact solution $q(x_j, t_n)$ at the point x_j and time t_n, and Δx and Δt are the spatial and temporal mesh size, respectively. Prove that the method converges under mesh refinements.

4. (30 points) Devise a fast solver for numerical approximation of the Poisson equation $\nabla^2 q = f$ over a circular domain in two space dimensions with the Dirichlet boundary condition. Algorithmic detail is required and so as the order of accuracy and convergence of the proposed scheme.