臺灣大學數學系
107 學年度下學期博士班資格考試題
科目：數值方法　　　　2019.02.22

02/21, 2019
Total scores: 100 points

1. (20 points) Let $I(f)$ be a definite integral defined by

$$I(f) = \int_0^1 f(x)\,dx,$$

and consider the quadrature formula

$$Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f^{'}(0)$$

for approximation of $I(f)$. Determine the coefficients α_j for $j = 1, 2, 3$ in such a way that Q has the degree of exactness $r = 2$.

Here the degree of exactness r is to find r such that

$$Q(x^k) = I(x^k) \quad \text{for} \quad k = 0, 1, \ldots, r, \quad \text{and} \quad Q(x^j) \neq I(x^j) \quad \text{for} \quad j > r.$$

2. (20 points) Consider the initial-value problem for the ordinary differential equation in the form:

$$\frac{du}{dt} = f(t, u(t)) \quad \text{with} \quad u(t_0) = u_0, \quad t > t_0, \tag{1a}$$

where $f(t, u(t))$ satisfies a suitable Lipschitz continuous condition with respect to u for the existence and uniqueness of the solution. Suppose that we want to use a two-step linear multistep method of the form

$$U_{n+2} + AU_{n+1} + BU_n = hCf_{n+1} \tag{1b}$$

for solving (1a) approximately, where U_n and f_n are the approximate solution of u and f at time t_n, respectively; h is the time step.

(a) (10 points) For what values of A, B, and C is the method (1b) with $O(h^2)$ truncation error ?

(b) (10 points) Assess the zero-stability of the method found in part (a).

3. (20 points) Consider the forward in time and central in space finite-difference method of the form

$$U_j^{n+1} = U_j^n + \frac{Dk}{h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right), \qquad j = 1, 2, \ldots, m, \tag{2}$$

for solving the heat equation

$$u_t = Du_{xx}, \quad D \in \mathbf{R}_+, \quad t > 0$$

with the initial condition

$$u(x, 0) = u_0(x)$$

over a periodic domain of length one: $x \in [0, 1]$. Here $U_j^n \approx u(x_j, t_n)$ is the numerical approximation of the solution $u(x, t)$ at the the spatial and temporal locations $x_j = jh$ and $t_n = nk$, respectively; $h = 1/m$ denotes the mesh size and k denotes the time step.

(a) (10 points) Show that the method (2) is consistent and stable under the condition $\mu = Dk/h^2 \leq 1/2$, and hence is convergent as the mesh is refined.

(b) (10 points) Show that the method (2) satisfies the discrete maximum principle

$$\|U^n\|_\infty \leq \|U^0\|_\infty \quad \forall\, n \geq 0$$

under the constraint $\mu = Dk/h^2 \leq 1/2$, where $U^n = \left[U_1^n, U_2^n, \ldots, U_m^n\right]^T$.

4. (40 points) Let z be an $m \times 1$ vector, $v = \|z\|_2 e_1 - z$, and $P = I - 2\frac{vv^T}{v^T v}$.

(a) (8 points) Show that P is symmetric and orthogonal.

(b) (8 points) Show that $Pz = \|z\|e_1$ and give a geometric explaination of the operation $Pz = \|z\|e_1$.

(c) (10 points) Write a pseudocode that applies the fact $Pz = \|z\|e_1$ to perform a QR factorization of an $m \times n$ matrix where $m > n$.

(d) (7 points) Estimate the computational complexity of the pseudocode in part (c).

(e) (7 points) Exaplain how you can apply a QR factorization to solve a general linear least square problem and a linear system problem.