臺灣大學數學系
九十八學年度上學期博士班資格考試題
科目：幾何與拓樸

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I. Let ∇ be the Levi-Civita connection on a Riemannian n-manifold M with a metric g_{ij} defined by

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle, i, j = 1, ...n.$$

Define the Christoffel symbol Γ_{ij}^k by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}, \quad i, j, k = 1, ..n.$$

and the Riemannian curvature tensor $R^m{}_{ijl}$ by

$$Rm(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^l} = R^m{}_{ijl} \frac{\partial}{\partial x^m} \quad \text{and} \quad R_{ijkl} = g_{mk} R^m{}_{ijl}.$$

Here

$$Rm(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^l} = \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l}.$$

Finally we define the Ricci curvature tensor and scalar curvature by

$$R_{ij} = g^{kl} R_{ikjl} \quad \text{and} \quad R = g^{ij} R_{ij}.$$

(5%) (i) Show that

$$\Gamma_{ij}^k = \Gamma_{ji}^k.$$

(10%) (ii) Show that

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(g_{lj,i} + g_{il,j} - g_{ij,l} \right). \text{ Here } \quad g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}.$$

(10%) (iii) Show that

$$R^m{}_{ijl} = \frac{\partial}{\partial x^i} \Gamma_{jl}^m - \frac{\partial}{\partial x^j} \Gamma_{il}^m + \Gamma_{in}^m \Gamma_{jl}^n - \Gamma_{jn}^m \Gamma_{il}^n.$$

(10%) (iv) Show that

$$\nabla_{\frac{\partial}{\partial x^i}} g_{jk} = 0$$

for all i, j, k.

(10%) (v) Show that

$$\nabla_{\frac{\partial}{\partial x^i}} R^i{}_j = \frac{1}{2} \nabla_{\frac{\partial}{\partial x^j}} R.$$

(10%) (vi) Suppose that for some smooth function ρ, we have

$$R_{ij} = \rho g_{ij}$$

on the whole manifold M. Show that ρ is constant and

$$\rho = \frac{R}{n}, \quad n > 2.$$

II. Let $(\mathbf{R}^2, \mathbf{g(t)})$ be a complete Riemannian surface with

$$\mathbf{g(t)} = \frac{dx^2 + dy^2}{e^{4t} + x^2 + y^2}, \quad dx^2 = dx \otimes dx.$$

(10%) (i) Show that in polar coordinates (r, θ), we may rewrite

$$\mathbf{g(0)} = ds^2 + \tanh^2 s d\theta^2, \quad s = \log(r + \sqrt{1 + r^2})$$

(10%) (ii) Show that the scalar curvature of $(\mathbf{R}^2, \mathbf{g(0)})$

$$R_0 = \frac{4}{1 + r^2}.$$

(10%) (iii) Find 1-parameter group of conformal diffeomorphisms $\varphi_t : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$g(t) = \varphi_t^* g(0).$$

III. Let $\mathbf{M}$ be an oriented differentiable n-manifold and $H^p_{deR}(\mathbf{M}, \mathbf{R})$ be the *pth* de Rham cohomology group. Show that

(10%) (i) If $\mathbf{M}$ is a closed manifold, then

$$\dim(H^p_{deR}(\mathbf{M}, \mathbf{R})) < \infty.$$

(5%) (ii) If $\mathbf{M} = \mathbf{R}^n$, then

$$H^p_{deR}(\mathbf{M}, \mathbf{R}) = 0$$

for all $p > 0$.