

國立臺灣大學數學系
九十六學年度上學期博士班資格考試題
科目：幾何與拓撲

2007.09

Answer FOUR questions.
There are five questions in total.
The questions carry equal weight.

All manifolds and related concepts should be assumed to be smooth.

1. Define a Riemannian metric g and the Levi-Civita connection D on a manifold. Prove that there exists a unique Levi-Civita connection. Let T be a differential one-form, and X any smooth vector field. Compute in local coordinates the covariant derivative of T with respect to D along X . Show that the covariant differentiation of the metric tensor g with respect to D vanishes identically. Define the curvature tensor in the above context, and prove the first Bianchi identity to be satisfied by the curvature tensor.
2. State the defining properties of the exterior derivative "d" for the differential forms on a manifold. Show that the exterior derivative d exists, and is uniquely determined. Let T be a differential one-form, and X, Y any smooth vector fields. Write a formula for $dT(X, Y)$, and prove this formula. Define de Rham cohomology and show how the cohomology classes can be related to harmonic forms arising from the Laplace operators on the manifold (suppose the manifold is equipped with a Riemannian metric g).
3. Explain what is meant by a geodesic on a Riemannian manifold M and deduce a system of ordinary differential equations satisfied by geodesics in local coordinates. Define the notion of geodesic completeness. What's the relation between the notions that M is geodesically complete and that M is complete as a metric space? Sketch a proof of your answer.
4. Let M be a compact manifold. Is it true that the fundamental group of M is isomorphic to its first homology group with integral coefficients? Assume that M is simply-connected. Is it true that the 2nd homotopy group of M is isomorphic to the 2nd homology group with integral coefficients? Give the reasons for your answers.
5. Let L be a geodesic in a Riemannian manifold. If p, q are points along L , state what it means for q to be conjugate to p along L , and to be a cut point of p along L . Prove that the cut point along L occurs at or before the first conjugate point.