臺灣大學數學系
108 學年度上學期博士班資格考試題
科目：幾何與拓樸 2019.09.12

(1) [15%] Consider the orthogonal group:

$$\mathrm{O}(n) = \{g \in \mathrm{GL}(n;\mathbb{R}) \,:\, g^{-1} = g^T\} \,.$$

Show that $\mathrm{O}(n)$ is a differentiable manifold, and determine its dimension.

(2) [15%+10%+5%] On $\mathbb{R}^3$, consider the following metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z + \sin z\,\mathrm{d}x + \cos z\,\mathrm{d}y)^2 \,.$$

(a) Calculate the Riemann curvature tensor of $\mathrm{d}s^2$.

(b) Denote the 1-form $\mathrm{d}z + \sin z\,\mathrm{d}x + \cos z\,\mathrm{d}y$ by $\alpha$. Can you find a regular surface[1] $\Sigma$ passing through the origin, and $T_p\Sigma \subset \ker(\alpha|_p)$ for every $p \in \Sigma$? Justify your answer.

(c) Same question as (b) for the 1-form $\beta = \mathrm{d}z + z\,\mathrm{d}x$ : Can you find a regular surface $\Sigma$ passing through the origin, and $T_p\Sigma \subset \ker(\beta|_p)$ for every $p \in \Sigma$? Justify your answer.

(3) [10%+10%] On a Riemannian manifold $(M, g)$, a vector field $U$ is called a Killing vector field if it is infinitesimally an isometry, namely,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi_t^* g = 0$$

where $\varphi_t$ is the one-parameter family of diffeomorphism generated by $U$.

(a) For a Killing vector field $U$, show that

$$g(\nabla_X U, Y) + g(\nabla_X U, Y) = 0$$

for any two vector fields $X, Y$.

(b) Suppose that $U$ is a Killing vector field, and $\gamma$ is a geodesic. Prove that $U|_\gamma$ is a Jacobi field.

[1]The regular surface here is assumed to have no boundary, and may not be compact.

(4) [5%+15%] Let $(M, g)$ be a connected Riemannian manifold, with $\dim M \geq 3$.

(a) State the second Bianchi identity for the Riemann curvature tensor.

(b) Suppose that its Ricci curvature is proportional to the metric tensor. Namely, there exists $f \in \mathcal{C}^\infty(M; \mathbb{R})$ such that

$$\operatorname{Ric}(X, Y) = f(p)\, g(X, Y)$$

for any $p \in M$, and $X, Y \in T_pM$. Prove that $f$ must be a constant function. (Hint: In terms of coordinate, the condition reads $R^{\ell}{}_{i\ell j} = f\, g_{ij}$. Taking covariant derivative in $\partial_k$ gives $R^{\ell}{}_{i\ell j;k} = (\partial_k f)\, g_{ij}$.)

(5) [9%+6%] Let $\Sigma$ be a closed (compact without boundary), oriented surface.

(a) Does there always exist a smooth map, $F : \Sigma \to S^2$, from $\Sigma$ to the 2-sphere, such that $F$ is essential (i.e. $F$ is not homotopic to a constant map)? Justify your answer.

(b) Consider the same question, but replacing the codomain $S^2$ by the 2-torus $T^2$.