臺灣大學數學系
107 學年度下學期博士班資格考試題
科目：幾何與拓樸　　　　2019.02.22

1. [10%+5%]

   (a) Suppose that (M, g) is a 3-dimensional Riemannian manifold which is Ricci flat. Does it have to be flat? If your answer is yes, give a proof. If your answer is no, give a counter-example.

   (b) What if M is 4-dimensional Ricci flat manifold? Explain the reason breifly.

2. [20%+10%] Let c be a non-negative constant. Consider the following metric on $\mathbb{R}^n$:

$$g_c = \frac{\sum_{j=1}^n (\mathrm{d}x^j)^2}{\left(1 + \frac{c}{4}\sum_{j=1}^n (x^j)^2\right)^2}.$$

   (a) Calculate the sectional curvatures of $(\mathbb{R}^n, g_c)$.

   (b) For which values of c is $(\mathbb{R}^n, g_c)$ complete? Justify your answer.

3. [10%+10%] Let $F : (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion. Namely, F is a submersion, and

$$\mathrm{d}F : \ker(\mathrm{d}F)^{\perp} \subset T_p\tilde{M} \to T_{F(p)}M \quad \text{is an isometry for every } p \in \tilde{M}.$$

   (a) Show that F shortens distances.

   (b) Prove that if $(\tilde{M}, \tilde{g})$ is a complete metric space, so is (M, g).

4. [10%+10%] Let (M, g) be a Riemannian manifold, and let f be a smooth function on M. Suppose that $U, V \in T_pM$; the Hessian of f in (U, V) is defined by

$$(\mathrm{Hess}^M f)(U, V) = U(\tilde{V}(f)) - (\nabla_U \tilde{V})(f)$$

   where $\tilde{V}$ is any smooth extension of V. One can check that it does not depend on the extension of V, and is symmetric in U, V. The Laplacian of f is defined to be the trace of the Hessian, $\Delta^M f = \mathrm{tr}_g \mathrm{Hess}^M f$.

   (a) Consider the unit sphere in the Euclidean space, $\mathbf{S}^n \subset \mathbb{R}^{n+1}$, with the induced metric. Consider the restriction of the standard coordinate function, x^j for $j = 1, \ldots, n+1$. Calculate $\mathrm{Hess}^{\mathbf{S}^n} x^j$.

   (b) Suppose that $\Sigma^k \subset \mathbf{S}^n$ is a minimal submanifold. Find $\Delta^{\Sigma} x^j$.

5. [5%+10%] Let M be a smooth 4-dimensional manifold. A symplectic form is a 2-form ω on M such that $\mathrm{d}\omega = 0$, and $\omega \wedge \omega$ is a nowhere vanishing 4-form.

   (a) Construct a symplectic form on $\mathbb{R}^4$.

   (b) Show that there are no symplectic forms on the 4-sphere, $\mathbf{S}^4$.