國立臺灣大學數學系
103學年度第2學期博士班資格考試題
科目：幾何與拓樸

2015.03.06

(1) (15%) Let $f(r)$ be a positive smooth function on $(0,\infty)$. Consider the following metric on $(0,\infty)\times S^2$:

$$g = (\mathrm{d}r)^2 + f(r)^2 g_{S^2} = (\mathrm{d}r)^2 + f(r)^2\left((\mathrm{d}\phi)^2 + (\sin\phi\,\mathrm{d}\theta)^2\right) .$$

Compute the Ricci curvature of g.

(2) (20%) Let (M,g) be a Riemannian manifold whose sectional curvature is greater than 1. Suppose that $\gamma(t):[0,1]\to M$ is a smooth curve with unit speed, and $V(t)$ is a nonzero parallel vector field along γ. For $s\in(-\epsilon,\epsilon)$, consider the curve $\gamma_s(t):[0,1]\to M$ defined by $\exp_{\gamma(t)}(sV(t))$. For s sufficient small, compare the length of $\gamma_s(t)$ and $\gamma(t)$. Explain your answer. (Hint: For any $t_0\in[0,1]$. Consider the vector field $\frac{\partial}{\partial t}\big|_{t=t_0}\gamma_s$ along the s-curve $\gamma_s(t_0)$.)

(3) Let (M,g) be a Riemannian manifold, and V be a vector field on M. It is called a *Killing vector field* if it preserves the metric, i.e. $L_V g = 0$.

(a) (10%) Show that V is a Killing vector field if and only if $V_{j;k}+V_{k;j}=0$ in terms of a local coordinate $\{x^j\}$. Here, $V_{j;k}$ means the $\frac{\partial}{\partial x^j}$-component of $\nabla_{\frac{\partial}{\partial x^k}}V$. Namely, $\nabla_{\frac{\partial}{\partial x^k}}V = \sum_j V_{j;k}\frac{\partial}{\partial x^j}$.

(b) (5%) Let $\gamma(t):(-\epsilon,\epsilon)\to M$ be a geodesic. Prove that $g(V,\gamma')$ is a constant (along γ) if V is a Killing vector field. (Hint: Calculate $\frac{\mathrm{d}}{\mathrm{d}t}g(V,\gamma')$.)

(4) Let G be a Lie group, and $\langle\ ,\ \rangle$ be a *left-invariant* metric on G. Suppose that X,Y,Z,W are *left-invariant* vector fields on G. Prove the following formulae.

(a) (7%) The Levi-Civita connection is given by

$$\langle\nabla_X Y, Z\rangle = \frac{1}{2}\left(\langle[X,Y],Z\rangle - \langle[X,Z],Y\rangle - \langle[Y,Z],X\rangle\right) .$$

(b) (6%) The Riemann curvature tensor is given by

$$\langle R(X,Y)Z,W\rangle = \langle\nabla_X Z,\nabla_Y W\rangle - \langle\nabla_Y Z,\nabla_X W\rangle - \langle\nabla_{[X,Y]}Z,W\rangle .$$

(c) (7%) If $\langle\ ,\ \rangle$ is *bi-invariant*, show that any left-invariant vector field is a Killing vector field. (Hint: A bi-invariant metric satisfies $\langle[X,Y],Z\rangle = \langle X,[Y,Z]\rangle$ for any left-invariant vector fields X,Y,Z.)

(5) (15%) Consider the standard n-sphere $S^n = \{(x^1, x^2, \cdots, x^{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{j=1}^{n+1}(x^j)^2 = 1\}$ with the metric induced from the standard metric of $\mathbb{R}^{n+1}$. Let $M \subset S^n$ be a k-dimensional submanifold. Prove that M is a minimal submanifold of S^n if and only if the coordinate functions $x^1, x^2, \cdots, x^{n+1}$ are eigenfunctions of M with eigenvalue k. That is to say,

$$-\Delta_M x^j = k\, x^j$$

for $j \in \{1, 2, \ldots, n+1\}$.

(Hint: Here, Δ_M is the Laplace–Beltrami operator of M with the induced metric g. Let f be a smooth function on M. The Hessian of f with respect to g is a $(0, 2)$-tensor defined by

$$\mathrm{Hess}_M(f)(X, Y) = \nabla_X(\nabla_Y f) - (\nabla_{\nabla_X Y})(f)$$

where ∇ is the Levi-Civita connection of g. The Laplace–Beltrami operator acting on f, $\Delta_M f$, is the traces of $\mathrm{Hess}_M(f)$ with respect of g.

If the function f is defined on $\mathbb{R}^{n+1}$, $\mathrm{Hess}_{\mathbb{R}^{n+1}}(f)$ is related to $\mathrm{Hess}_{S^n}(f)$, and $\mathrm{Hess}_{S^n}(f)$ is also related to $\mathrm{Hess}_M(f)$.)

(6) Denote by Σ_g a closed, orientable surface of genus g. Suppose that $f : \Sigma_g \to \Sigma_h$ is a continuous map from a surface of genus g to a surface of genus f.

(a) (5%) Define the *degree* of the map f. There may be more many ways to do it, and you only have to provide one definition.

(b) (10%) If $g < h$, does that exist a continuous map $f : \Sigma_g \to \Sigma_h$ with *nonzero* degree? Explain your answer. (Hint: The cohomology constitutes a ring.)