臺灣大學數學系

102 學年度下學期博士班資格考試題

科目：幾何與拓樸 2014.02.20

1. (20%)

(a) Let M be a closed differentiable manifold of dimension n. Show that M can be embedded into $\mathbf{R}^{\mathbf{N}}$ for some positive integer $\mathbf{N}$.

(b) Show that the projective plane $\mathbf{R}P^2$ can be embedded into $\mathbf{R}^4$.

2. (20%) Let (Σ, g) be a smooth Riemann surface with rotationally symmetric metric $g = dr^2 + f^2(r)d\theta^2$ for some positive smooth function $f(r), 0 < r$.

(a) Compute

$$K = -\frac{f''}{f},$$

where K is the Gaussian curvature.

(b) In particular,

$$K = \begin{cases} 1 & , \quad f(r) = \sin r, \\ 0 & , \quad f(r) = r, \\ -1 & , \quad f(r) = \sinh r. \end{cases}$$

3. (20%)

Let (M, g) be a complete Riemannian manifold of dimension n with the Ricci curvature bounded from below by $(n-1)K$, where $K > 0$.

(a) Show that

$$diam(M) \leq \frac{\pi}{\sqrt{K}}.$$

(b) Show that M is compact and $\pi_1(M) < \infty$.

4. (20%)

Let $f : \Sigma \to \mathbf{R}^3$ be an isometric immersion of a smooth closed orientable Riemann surface Σ into $\mathbf{R}^3$. We define the Willmore energy

$$W(f) = \int_\Sigma H^2 dA,$$

where $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ is the mean curvature and κ_1, κ_2 are the principal curvatures.

(a) Show that

$$\int_{\Sigma} K^+ dA \geq 4\pi,$$

where $K^+ = \max\{K, 0\}$, $\Sigma^+ = \Sigma|_{K^+}$ and $K = \kappa_1 + \kappa_2$ is the Gauss curvature.

(b) Show that

$$W(f) \geq 4\pi.$$

Moreover, $W(f) = 4\pi$ if and only if Σ is embedded as a round sphere in $\mathbf{R}^3$, i.e. $\kappa_1 = \kappa_2$ at every point.

5. (20%)

(a) Let (M, g) be a closed, orientable Riemannian manifold of dimension n. Show that all de Rham cohomology groups $H^p_{dR}(M, \mathbf{R}), 0 \leq p \leq n$, are finite dimensional.

(b) Let $w_1, w_2, ..., w_n \in \mathbf{R}^n$ be linearly independent. We consider z_1, z_2 in $\mathbf{R}^n$ as equivalent if there are $m_1, m_2, ..., m_n \in \mathbf{Z}$ with $z_1 - z_2 = \sum_{i=1}^n m_i w_i$. Let π be the projection mapping z to its equivalence class. Now we consider the n-dimensional torus $\mathbf{T}^n := \pi(\mathbf{R}^n)$. Compute all p-th Betti number $b_p(\mathbf{T}^n)$ for $0 \leq p \leq n$.