臺灣大學數學系
99 學年度上學期博士班資格考試題
科目：離散數學　　2010.09.17

1. (20%) Suppose G_n is the graph with vertex set $V(G_n) = \{v_0, v_1, v_2, \ldots, v_n\}$ and edge set $E(G_n) = \{v_0v_i : 1 \le i \le n\} \cup \{v_iv_{i+1} : 1 \le i \le n-1\}$. Determine the number $\tau(G_n)$ of spanning trees of G_n.

2. (20%) Suppose M is an $n \times n$ matrix of non-negative real numbers in which each row or each column sums up to 1. Prove that there is a positive integer r, r positive reals $c_1, c_2, \ldots, c_r$, and r permutation matrices $P_1, P_2, \ldots, P_r$ such that $M = \sum_{i=1}^{r} c_iP_i$.

3. (20%) (a) Prove that $\chi(G) \le \Delta(G) + 1$ for any graph G.

   (b) Prove that $\chi(G) \le \Delta(G)$ for any connected graph G that is neither a complete graph nor an odd cycle.

4. (20%) Consider the Ramsey number $R(m, n)$ which is the minimum number k such that for every 2-edge coloring of K_k there is always a monochromatic copy of K_m or K_n.

   (a) Prove that $R(m, n) \le R(m-1, n) + R(m, n-1) + 1$. Use this to derive an upper bound for $R(m, n)$ in terms of m and n.

   (b) Derive a lower bound for $R(n, n)$.

5. (20%) The thickness of a graph G, denoted by $t(G)$, is the minimum k such that the edges of G can be decomposed into k planar graphs.

   (a) Determine $t(K_{4,4})$.

   (b) Determine $t(K_{5,5})$.

   (c) Prove that $t(K_n) \ge \lceil (n+2)/6 \rceil$.

   (d) Prove that if $n \ne 9, 10$, then $t(K_n) = \lceil (n+2)/6 \rceil$.

   (e) What are the values of $t(K_9)$ and $t(K_{10})$? Why?