臺灣大學數學系

九十七學年度上學期博士班資格考試題

科目：離散數學　　2008.09.18

1. Let S be an n-element set, and let $\{A_1, A_2, \ldots, A_n\}$ be n distinct subsets of S. Prove that S has an element x such that the sets $A_1 \cup \{x\}, A_2 \cup \{x\}, \ldots, A_n \cup \{x\}$ are distinct.

2. Let G be an X, Y-bipartite graph having a matching that saturates X.

   (a) Let S and T be subsets of X such that $|N(S)| = |S|$ and $|N(T)| = |T|$. Prove that $|N(S \cap T)| = |S \cap T|$.

   (b) Prove that X has some vertex x such that every edge incident to x belongs to some maximum matching.

3. A k-edge-connected graph G is minimally k-edge-connected if for every $e \in E(G)$ the graph $G - e$ is not k-edge-connected. Prove that $\delta(G) = k$ when G is minimally k-edge-connected.

4. An acyclic orientation of a loopless graph is an orientation having no cycle. For each acyclic orientation D of G, let $r(D) = \max_C \lceil a/b \rceil$, where C is a cycle in G and a, b count the edges of C that are forward in D or backward in D, respectively. Fix a vertex $x \in V(G)$, and let W be a walk in G beginning at x. Let $g(W) = a - b \cdot r(D)$, where a is the number of steps along W that are forward edges in D and b is the number of backward in D. For each $y \in V(G)$, let $g(y)$ be the maximum of $g(W)$ such that W is an x, y-walk (assume that G is connected).

   (a) Prove that $g(y)$ is finite and thus well-defined, and use $g(y)$ to obtain a proper $1 + r(D)$-coloring of G. Thus G is $1 + r(D)$-colorable.

   (b) Prove that $\chi(G) = \min\{1 + r(D) : D \text{ is an acyclic orientation of } G\}$.

5. The kth power of a graph G is the graph G^k with vertex set $V(G)$ and edge set $\{uv : 1 \leq d_G(u, v) \leq k\}$.

   (a) Suppose that $G - x$ has at least three nontrivial components in each of which x has exactly one neighbor. Prove that G^2 is not Hamiltonian.

   (b) Prove that the cube of each connected graph with at least three vertices is Hamiltonian.