

國立臺灣大學數學系
九十六學年度下學期博士班資格考試題
科目：離散數學

2008.02

- (8%) (a) In how many ways can the integers $1, 2, \dots, n$ be permuted such that no integer will be in its natural position? In other words, count the number of permutations $(\pi_1, \pi_2, \dots, \pi_n)$ of the integers $1, 2, \dots, n$ with $\pi_i \neq i$ for $1 \leq i \leq n$.

(8%)(b) In how many ways can the integers $1, 2, 3, 4, 5, 6, 7, 8, 9$ be permuted such that no odd integer will be in its natural position?
- (16%) Suppose A_1 and A_2 are two Latin squares of order n . Let $a_{ij}^{(1)}$ and $a_{ij}^{(2)}$ ($1 \leq i \leq n, 1 \leq j \leq n$) denote the entries in the i th row and the j th column in A_1 and A_2 , respectively. The two Latin squares A_1 and A_2 are said to be orthogonal if the n^2 pairs $(a_{ij}^{(1)}, a_{ij}^{(2)})$ ($1 \leq i \leq n, 1 \leq j \leq n$) are all distinct.

(a) Prove that there are at most $n - 1$ Latin squares in a set of pairwise orthogonal Latin squares of order n .

(b) Prove that for $n \geq 3$ and $n = p^\alpha$, where p is a prime number and α a positive integer, there is a set of $n - 1$ pairwise orthogonal Latin squares of order n .
- (17%) For $n \geq 3$, determine the minimum number of edges in a connected graph of n vertices in which every edge belongs to a triangle.
- (10%) (a) Let G be a k -regular graph of even order that remains connected when any $k - 2$ edges are deleted. Prove that G has a perfect matching.

(7%) (b) For each odd integer k greater than 1, construct a graph G with no perfect matching that is k -regular and remains connected when any $k - 3$ edges are deleted.
- (17%) Let v be a vertex of a 2-connected graph G . Prove that v has a neighbor u such that $G - u - v$ is connected.
- (17%) Let G be a k -chromatic graph with girth 6 and order n . Construct a new graph G' as follows. Let T be an independent set of kn new vertices. Take $\binom{kn}{n}$ pairwise disjoint copies of G , one for each way to choose an n -set $S \subset T$. Add a matching between each copy of G and its corresponding n -set S . Prove that the new graph G' has chromatic number $k + 1$ and girth 6.