國立臺灣大學數學系

九十六學年度上學期博士班資格考試題

科目：離散數學

2007.09

1. (16%) Recall that $\lfloor x \rfloor$ is the largest integer less than or equal to x. Consider the following recurrence:

$$K_0 = 1; \quad K_{n+1} = 1 + \min\{2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}\} \text{ for } n \geq 0.$$

   (a) Prove or disprove that $K_n \geq n$ for any nonnegative integer n.

   (b) Prove or disprove that $K_n \leq 2n$ for any integer $n \geq 2$.

2. (16%) Prove that there is a function $f(n)$ on the natural numbers with the property that, if the numbers $\{1, 2, \ldots, f(n)\}$ are partitioned into n classes, then there are two numbers x and y such that x, y and $x + y$ all belong to the same class. (In other words, the numbers $\{1, 2, \ldots, f(n)\}$ cannot be partitioned into n 'sum-free sets'.)

3. (16%) Let G be a graph of girth 5. Prove that if every vertex of G has degree at least k, then G has at least $k^2 + 1$ vertices. For $k = 2$ and $k = 3$, find one such graph with exactly $k^2 + 1$ vertices.

4. (16%) Prove that if $T_1, T_2, \ldots, T_k$ are pairwise-intersecting subtrees of a tree T, then T has a vertex that belongs to all of $T_1, T_2, \ldots, T_k$.
(This result is called the Helly property of trees.)

5. (16%) An algorithm to greedily build a large independent set S iteratively selects a vertex of minimum degree in the remaining graph, adds it to S, and deletes it and its neighbors from the graph. Prove that this algorithm produces an independent set of size at least $\sum_{v \in V(G)} \frac{1}{d_G(v)+1}$ in a simple graph G.

6. (20%) For an ordering $v_1, v_2, \ldots, v_n$ of the vertex set $V(G)$ of a graph G, let $\chi(G; v_1, v_2, \ldots, v_n)$ be the number of colors needed if a greedy coloring algorithm is applied using the ordering $v_1, v_2, \ldots, v_n$. Define

$$A(G) = \{\chi(G; v_1, v_2, \ldots, v_n) : v_1, v_2, \ldots, v_n \text{ is an ordering of } V(G)\},$$

   $\chi_{\max}(G)$ is the maximum number in $A(G)$ and $\chi_{\min}(G)$ is the minimum number in $A(G)$.

   (a) Determine $A(P_4)$, where P_4 is the path of 4 vertices.

   (b) Prove that $\chi(G) = \chi_{\min}(G)$ for any graph G.

   (c) Prove that $\chi_{\max}(G) \leq \Delta(G) + 1$ for any graph G.

   (d) Prove that $\chi_{\min}(G) = \chi_{\max}(G)$ if G is P_4-free. Characterize graphs G in which $\chi_{\min}(H) = \chi_{\max}(H)$ for all induced subgraphs H of G.

   (e) For any fixed k, is there any graph G such that $\chi_{\max}(G) - \chi_{\min}(G) \geq k$?