# 臺灣大學數學系111學年度第1學期博士班一般資格考試

科目：離散數學 2022.09.12

1. (20%) Define the Turán number $ex(n, F)$ of a graph $F$, and prove that

$$ex(n, K_{k+1}) = \left(1 - \frac{1}{k} + o(1)\right)\binom{n}{2}.$$

What do we know about the Turan number of general graphs $F$?

2. (20%) A $k$-chain is a set of $k$ sets satisfying $F_1 \subset F_2 \subset \cdots \subset F_k$. Let $\mathcal{F} \subset 2^{[n]}$ be a family of subsets of $[n]$ that does not contain a $k$-chain. Prove that

$$\sum_{F\in\mathcal{F}} \binom{n}{|F|}^{-1} \le k - 1,$$

and use this to determine the largest possible size of a $k$-chain-free family of subsets of $[n]$.

3. (20%) State Ramsey's Theorem and prove exponential lower and upper bounds on the diagonal Ramsey number $R(t)$.

4. (20%) Let $G$ be a graph on $n$ vertices with $m$ edges, and suppose we have a drawing of $G$ in the plane. A *crossing* in the drawing is a pair of vertex-disjoint edges for which the corresponding paths in the drawing intersect.

(a) State Euler's Formula for planar graphs and deduce that if $m \ge 4n$, then the drawing must contain a crossing.

(b) Improve the bound by showing that the drawing must in fact contain at least $m - 4n$ crossings.

(c) By considering, for some appropriate choice of $p$, a random subgraph of $G$, where each vertex is retained independently with probability $p$, show that the drawing must have at least $\frac{m^3}{1000n^2}$ crossings.

5. (20%) A *Steiner Triple System* over the ground set $[n]$ is a 3-uniform family $\mathcal{F} \in \binom{[n]}{3}$ such that every pair in $\binom{[n]}{2}$ is contained in exactly one set $F \in \mathcal{F}$.

(a) Give examples of Steiner Triple Systems for $n = 7$ and $n = 9$.

(b) By considering the sets containing an element $x \in [n]$, as well as the total number of sets, prove that if a Steiner Triple System on $[n]$ exists, then $n \equiv 1$ or $3$ modulo $6$.