臺灣大學數學系
99 學年度下學期博士班資格考試題
科目：代數　　　　2011.02.25

(1) (15 %) Let k be a field, and $k[[x]]$ be the set of all formal power series $\sum_{i=0}^{\infty} a_i x^i$, with coefficients $a_i \in k$. Define operations on $k[[x]]$ by:

$$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i,$$

$$(\sum_{i=0}^{\infty} a_i x^i) \cdot (\sum_{i=0}^{\infty} b_i x^i) = \sum_{i=0}^{\infty} (\sum_{j=0}^{i} a_j b_{i-j}) x^i.$$

Show that $k[[x]]$ is a principal ideal domain under these operations. Furthermore show that there is only one maximal ideal inside this ring.

(2) (15 %) Let $\mathrm{Mat}_n(\mathbb{Q})$ be the ring consisting of $n \times n$ square matrices with entries from $\mathbb{Q}$. Prove that any automorphism of this ring must be of the form $M \mapsto AMA^{-1}$, for some invertible matrix $A \in \mathrm{Mat}_n(\mathbb{Q})$.

(3) (15 %) Let M be any given $n \times n$ matrix having entries from field k, with characteristic polynomial $P(x) \in k[x]$. Give a proof of the following identity (Cayley-Hamilton) :

$$P(M) = 0.$$

(4) (15 %) Let $\mathbb{F}_{32}$ be the finite field with 32 elements, $G := \mathrm{GL}_{32}(\mathbb{F}_{32})$ is the group of invertible 32×32 matrices having entries from $\mathbb{F}_{32}$. Show that the subgroup U of G consisting of upper trangular matrices having its diagonal entries all equal to 1 is a Sylow 2-subgroup of G. Then prove that any finite group of order 32 is isomorphic to a subgroup of this matrix group U (you may use Sylow's theorems to prove this statement).

(5) (10%) Use Galois theory to compute the Galois group of the polynomial $x^5 - 5$ over $\mathbb{Q}$.

(6) (15 %) Let $\mathcal{P}$ ba a given maximal ideal in the ring $\mathbb{Z}[\sqrt{-1}]$. Show first that $\mathcal{P} \cap \mathbb{Z} = (p)$, where p is a prime number. Denote by $\overline{\mathcal{P}}$ the set of complex conjugates of elements in $\mathcal{P}$. Then verify that if $\mathcal{P}_1, \mathcal{P}_2$ are two maximal ideals in $\mathbb{Z}[\sqrt{-1}]$ satisfying $\mathcal{P}_i \cap \mathbb{Z} = (p)$, for $i = 1, 2$, then either $\mathcal{P}_1 = \mathcal{P}_2$, or $\overline{\mathcal{P}_1} = \mathcal{P}_2$ (you may apply Chinese remainder theorem to the ring $\mathbb{Z}[\sqrt{-1}]$).

(7) (15 %) Let $V := \mathbb{C}^n$ be the n-dimensional complex vector space equipped with the hermitian form

$$(u, v) := \sum_{i=1}^{n} u_i \bar{v}_i, \quad u, v \in \mathbb{C}^n.$$

Recall that a linear transformation $T : V \to V$ is said to be self-adjoint if it satisfies:

$$(T(u), v) = (u, T(v)), \quad \forall u, v \in V.$$

Given a set S of self-adjoint transformations on V which are mutually commuting, i.e. $T_1 \circ T_2 = T_2 \circ T_1$ if $T_1, T_2 \in S$. Prove that this space V must have a basis consisting of elements which are eigenvectors for all transformations in S.