臺灣大學數學系

九十八學年度上學期博士班資格考試題

科目：代數 2009.09.18

(1) (20%)Classify groups of order $4p$, where p is a prime greater than 3 and $p \equiv 3 \pmod 4$. Are they always solvable? (Justify your answer)

(2) (30%)Let R be a commutative ring with identity and M a finitely generated R-module.

(a) Assume that $M \otimes_R \kappa(m) = 0$ for every maximal ideal m, where $\kappa(m)$ is the residue field of the local ring R_m. Show that $M = 0$.

(b) Show that any submodule of a free module of finite rank over a PID is free and thus every finitely generated projective module over a PID is free.

(c) Show that if R is a Noetherian ring, then $R[[x]]$ is also a Noetherian ring.

(3) (20%)Determine the Galois group of $x^4 - 7$ over $\mathbb{Q}$ and $\mathbb{Q}[\sqrt{7}]$ respectively.

(4) (15%)Let $s_n := \frac{n(n+1)}{2}$ for $n \in \mathbb{N}$. Show that each positive integer can be written as a sum of finitely many s_n's.

(5) (15%)Let $SO(3,\mathbb{R}) := \{A \in GL(3,\mathbb{R}) | A^t A = I, \det(A) = 1\}$. Let A be a matrix in $SO(3,\mathbb{R})$. Show that A is similar to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ for some θ.