臺灣大學數學系
九十七學年度上學期博士班資格考試題
科目：代數 2008.09.19

Notations:
$\mathbb{R}$: the field of real numbers.
$\mathbb{C}$: the field of complex numbers.
$GL(n, F)$: the group of non-singular $n \times n$ matrices over the field F.
$SO(n, F) := \{A \in GL(n, F) | A^t A = I, \det(A) = 1\}$.

(1) (15 pts) Let

$$\Sigma := \{(x, y) | x = \pm 1, y = \pm 1\} \subset \mathbb{R}^2,$$
$$\Gamma := \{(x, y, z) | x = \pm 1, y = \pm 1, z = \pm 1\} \subset \mathbb{R}^3.$$

The (orientation-preserving) symmetries of Σ (resp. Γ) is defined to be $g \in SO(2, \mathbb{R})$ (resp. $SO(3, \mathbb{R})$) such that $g(\Sigma) = \Sigma$ (resp. $g(\Gamma) = \Gamma$).

Determine the group of symmetries of Σ and Γ.

(2) (20 pts) Show that for any finite group G, there is a Galois extension with Galois group isomorphic to G.

(3) (15 pts) Let R be a unique factorization domain. Show that $R[x]$ is a unique factorization domain.

(4) (20 pts) Let G be a non-abelian group of order 27 such that every non-identity element in it has order 3. Can G be embedded into $GL(2, \mathbb{C})$ as a subgroup? Explain why.

(5) (10 pts) Let

$$A = \prod_{i=1}^{n} a_i, \quad B = \prod_{j=1}^{m} b_j$$

be products of positive integers > 1. Suppose that the greatest common divisor h of any μ of the $\{a_i\}$ divides at least μ of the $\{b_j\}$. Show that A divides B.

(6) (20 pts) Let $R = \mathbb{C}[x, y]/(y^2 - x^3)$. Find all the prime ideals of R. For a prime ideal $\mathfrak{p}$ of R, one can consider its localization $R_{\mathfrak{p}}$. How many different isomorphic classes of $R_{\mathfrak{p}}$ are there? Explain why.