國立臺灣大學數學系
九十五學年度博士班資格考試試題
科目：代數

2007.06.01

(1) Solve the equation $x^2 + x + 47 \equiv 0 \pmod{7^3}$.

(2) Let

$$a = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, b = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, c = \frac{1}{2}\begin{bmatrix} -1+i & -1+i \\ 1+i & -1-i \end{bmatrix}, d = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix}$$

$\in GL_2(\mathbb{C})$ and let G be the subgroup generated by a, b, c, d.

(a) Find the order of G.

(b) Show that G has a homomorphic image isomorphic to the symmetric group S_4.

(3) Let K be a field which is not algebraically closed.

(a) For any m, show that there is a polynomial $g \in K[x_1, \ldots, x_m]$ such that the only solution of the equation $g = 0$ is $(0, 0, \ldots, 0)$.

(b) Let $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ and S be the set of solutions of the system of equations $f_1 = 0, \ldots, f_m = 0$. Show that there is a single polynomial p such that S is the set of solutions of the equation $p = 0$.

(c) Does the statement (b) hold for $K = \mathbb{C}$?

(4) Let D be a division ring of characteristic $p \neq 0$ and let G be a finite subgroup of the multiplicative group $D \setminus \{0\}$. Show that G is cyclic.

(5) Let $L = K(x, y)$ where x is transcendental over K and $x^2 + y^2 = 1$.

(a) Show that $L = K(t)$ for some t.

(b) Find the group of K-automorphisms of L.