# 臺灣大學數學系112學年度第1學期博士班一般資格考試

## 科目：代數

2023.09.06

**1.** Let $V \neq 0$ be a finite-dimensional vector space over a field $F$ and $T : V \to V$ be a linear transformation. Recall that a cyclic vector for $T$ is a vector $v$ in $V$ such that $\{v, Tv, T^2v, \ldots\}$ spans $V$.

(a) (**10 points.**) Suppose that every nonzero vector of $V$ is a cyclic vector for $T$. Prove that the characteristic polynomial of $T$ must be irreducible over $F$.

(b) (**10 points.**) Suppose that the characteristic polynomial of $T$ is irreducible over $F$. Prove that every nonzero vector $v$ in $V$ is a cyclic vector for $T$.

**2.** Let $p$ be a prime and $G$ be a finite $p$-group.

(a) (**5 points.**) Prove that the center $Z(G)$ of $G$ is nontrivial.

(b) (**5 points.**) Prove that if $G/Z(G)$ is cyclic, then $G$ is abelian.

(c) (**15 points.**) Assume that $p$ is an odd prime. Classify groups of order $p^3$ up to isomorphisms.

(d) (**5 points.**) Assume that $p$ is an odd prime. Can a non-abelian group of order $p^3$ be embedded into $\mathrm{GL}(2, \mathbb{C})$ as a subgroup? Explain your reasoning.

**3.** (**10 points.**) Let $R$ be a Noetherian integral domain. Prove that every nonzero element of $R$ can be factored into a product of irreducibles.

**4.** (a) (**5 points.**) Let $f(x)$ be a monic irreducible polynomial of degree 4 over $\mathbb{Q}$ such that the Galois group of $f$ over $\mathbb{Q}$ is cyclic of order 4. Prove that the discriminant of $f$ is not equal to the square of a rational number.

(b) (**20 points.**) Let $a$ be a squarefree integer with $a \neq 1$. Prove that $\mathbb{Q}(\sqrt{a})$ is contained in some cyclic extension of degree 4 of $\mathbb{Q}$ if and only if $a = b^2 + c^2$ for some $b, c \in \mathbb{Q}$. (*Hint for the "only if" direction*: Assume that $K/\mathbb{Q}$ is a cyclic extension of degree 4 such that $K$ contains $\mathbb{Q}(\sqrt{a})$. Let $\alpha \in K$ be a primitive element and $f(x)$ be its irreducible polynomial over $\mathbb{Q}$. What does Part (a) say about the discriminant of $f$?)

**5.** Let $R$ be a commutative ring with 1. Suppose that

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence of $R$-modules. Let $r$ and $s$ be elements of $R$ such that $R$ is generated by $r$ and $s$ and $rA = sC = 0$.

(a) (**5 points.**) Show that the map $C \to C$ given by $c \mapsto rc$ is an isomorphism.

(b) (**5 points.**) Show that the restriction of $g$ to $rB$ gives an isomorphism $rB \simeq C$.

(c) (**5 points.**) Show that $B \simeq A \oplus C$.