臺灣大學數學系110學年度第2學期博士班一般資格考試

科目：代數

2022.04.13

(1) (25%) Let p be a prime number. Show that a finite p-group has nontrivial center. Find a *non-commutative group* of order p^4, non-isomorphic to a direct product of smaller order groups.

(2) (25%) Let $\mathcal{O}$ be a commutative ring. Prove the follow.

(a) Suppose ideals in $\mathcal{O}$ satisfy the descending chain condition: if ideals

$$\mathcal{O} \supseteq I_1 \supseteq I_2 \supseteq \cdots I_k \supseteq \cdots,$$

then there exists some n such that $I_k = I_n$, for all $k \geq n$. Then $\mathcal{O}$ is Noetherian.

(b) If $\mathcal{O}$ is Noetherian, then the formal power series ring $\mathcal{O}[[x]]$ is also Noetherian.

(3) (25%) Let F be a field of characteristic 0 and $f(x) \in F[x]$ be a degree 4 (not necessary irreducible) polynomial. Let $K = F(\theta_1, \theta_2, \theta_3, \theta_4)$, where $\theta_1, ..., \theta_4$ are distinct roots of $f(x)$. Define

$$\xi = (\theta_1 + \theta_2)(\theta_3 + \theta_4).$$

Show that ξ satisfies a degree 3 polynomial $g(x) \in F[x]$ and if L denote the splitting field of $g(x)$ over F, then $K = L(\sqrt{\alpha}, \sqrt{\beta})$, for some $\alpha, \beta \in L$.

(4) (25%) Let F be a field and let $M_n(F)$ denote the ring of all $n \times n$ matrices over F. For a subset $S \subset M_n(F)$, let $\text{span}_F(S) \subset M_n(F)$ denote the linear F-subspace spanned by S. We say that S is *strong*, if every non-zero element in $\text{span}_F(S)$ has non-zero determinant. Let A be a matrix in $M_n(F)$. Prove that the set $S_A := \{I_n, A, A^2, ..., A^k, ...\}$ is strong if and only if the minimal polynomial of A is irreducible in $F[x]$, and in this case $\text{span}_F(S_A)$ is actually a subfield of $M_n(F)$.