臺灣大學數學系
110 學年度上學期博士班資格考試題
科目：代數

2021.09.24

(1) (25%) Let V be an n-dimensional vector space over a field F, with $n \geq 2$.

(a) Suppose F is a finite field of order q. Find proper subspaces $V_1, \ldots, V_{q+1}$ of V such that $\bigcup_{i=1}^{q+1} V_i = V$.

(b) Show that if F is an infinite field, then V can not be a union of finitely many proper subspaces.

(2) (25%). Let D be a nonzero integer and denote $R_D := \mathbb{Z}[\sqrt{D}]$.

(a) Show that if $D \equiv 1 \pmod 4$, then R_D is not a U.F.D.

(b) Find an integer $D \equiv 3 \pmod 4$ such that R_D is a U.F.D, also find an integer $D \equiv 3 \pmod 4$ such that R_D is not a U.F.D. Prove you assertions.

(3) (25%) Let F be a field and K/F a finite extension.

(a) Show that $K = F(\theta)$, for some $\theta \in K$, if and only if there are only finitely many distinct fields E satisfying $F \subset E \subset K$.

(b) Let K be the splitting field of a separable irreducible polynomial $f(x) \in F[x]$ and denote $G := \mathrm{Gal}(K/F)$. If for each $g \in G$, there is an $\alpha \in K$, $f(\alpha) = 0$, such that ${}^g\alpha = \alpha$, then $K = F$.

(4) (25%) Let A be a Noetherian integral domain. For an A-module M, denote $M_{tor} := \{m \in M \mid a \cdot m = 0, \text{ for some nonzero } a \in A\}$. M is said to be torsion free if $M_{tor} = \{0\}$.

(a) Show that A is a P.I.D. if and only if every torsion free finitely generated A-module is a free module.

(b) If A is a P.I.D. then the exact sequence

$$0 \longrightarrow M_{tor} \longrightarrow M \longrightarrow M/M_{tor} \longrightarrow 0$$

splits.