臺灣大學數學系

108 學年度上學期博士班資格考試題

科目：代數 2019.09.12

Notations:

$\mathbb{Z}$: the ring of integers.

$\mathbb{Q}$: the field of rational numbers.

(1) (30%)
  (a) Let p, q be prime numbers and let G be a group of order p^2q. Prove or disprove that G is solvable.
  (b) Classify groups of order $4p$, where p is a prime greater than 3 and $p \equiv 3 \pmod 4$.

(2) (20%)
  (a) Let $m \in \mathbb{Z}$ be square-free and let A be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{m}]$. Show that $A = \mathbb{Z}[(1+\sqrt{m})/2]$ if $m \equiv 1 \pmod 4$ and $A = \mathbb{Z}[\sqrt{m}]$ otherwise.
  (b) If R is Noetherian, then $R[[x]]$ is also Noetherian.

(3) (25%)
  (a) Show that a module is projective if and only if it is a direct summand of some free module.
  (b) Show that every module over a PID is injective if and only if it is divisible.
  (c) Let M be a flat R-module and $a \in R$ be not a zero-divisor. Show that if $ax = 0$ for some $x \in M$ then $x = 0$.

(4) (25%) Let $L = \mathbb{Q}(\cos \pi/9)$ and $K = \mathbb{Q}$.
  (a) Show that L is a splitting field of a separable polynomial $f(x)$ in $K[x]$.
  (b) Show that L/K is not an extension by radicals.
  (c) Find a Galois extension by radicals E/K such that $L \subset E$.
  (d) Find the Galois group $\mathrm{Gal}(E/K)$ of your example.