臺灣大學數學系
107 學年度上學期博士班資格考試題
科目：代數

2018.09.13

Notations:
$\mathbb{Z}$: the ring of integers.
$\mathbb{Q}$: the field of rational numbers.
$Z(G)$: the center of the group G.

(1) (25%) Let G be a non-abelian group of order p^3 with p a prime.
   (a) Show that $Z(G) \simeq \mathbb{Z}/p\mathbb{Z}$ and $G/Z(G) \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$,
   (b) Let H be a subgroup of order p^2. Show that $H \supset Z(G)$ and H is normal.
   (c) Show that if $a^p = 1$ for all $a \in G$, then there exists a normal subgroup $H \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.
   (d) Show that there are at most two non-abelian groups of order p^3.

(2) (25%)
   (a) Let I be the set of all non-units of $\mathbb{Z}[\sqrt{-1}]$. Is I an ideal of $\mathbb{Z}[\sqrt{-1}]$? (Justify your answer)
   (b) Show that $\mathbb{Z}[\sqrt{-1}]$ is a UFD.
   (c) Show that for any non-trivial ideal J of $\mathbb{Z}[\sqrt{-1}]$, the quotient ring $\mathbb{Z}[\sqrt{-1}]/J$ is a finite ring.
   (d) Determine all prime elements in $\mathbb{Z}[\sqrt{-1}]$. (Justify your answer)

(3) (25%)
   (a) Let R be a PID and M be a finitely generated R-module. Show that the following sequence is split exact:

$$0 \longrightarrow M_{\text{tor}} \longrightarrow M \longrightarrow M/M_{\text{tor}} \longrightarrow o$$

   where M_{tor} is the torsion submodule of M.
   (b) Let R be a local ring, m be the maximal ideal of R and $k = R/m$ be the residue field of R. Let M be a finitely generated R-module. Show that $\{x_1, \ldots, x_n\}$ is a minimal generating set of M if and only if the images $\bar{x_1}, \ldots, \bar{x_n}$ in M/mM form a basis for M/mM over k.
   (c) Show that any submodule of a free module of finite rank over a PID is free.

(4) (25%)
   (a) Determine the Galois group G of the splitting field L of $x^7 - 1$ over $\mathbb{Q}$ and the correspondence between subgroups of G and subextensions of $L/\mathbb{Q}$. (Justify your answer.)
   (b) Use the solvability of groups to explain why cubic polynomials and quartic polynomials over $\mathbb{Q}$ are solvable by radicals.
   (c) Is the polynomial $f(x) = x^5 - 4x + 2$ solvable by radicals over $\mathbb{Q}$? (Explain your answer.)