臺灣大學數學系
106 學年度上學期博士班資格考試題
科目：代數 2017.09.15

Problem 1 (20%). Let G be a group. For a map $\phi : G \to \mathbb{Q}$ and $g \in G$, define the new map $\phi^g : G \to \mathbb{Q}$ by

$$\phi^g(h) = \phi(gh) - \phi(h).$$

Show that $(\phi^{g_1})^{g_2} = 0$ for all $g_1, g_2 \in G$ if and only if there is a group homomorphism $\psi : G \to \mathbb{Q}$ such that $\phi - \psi$ is a constant (i.e., $\phi(h_1) - \psi(h_1) = \phi(h_2) - \psi(h_2)$ for all $h_1, h_2 \in G$).

Problem 2 (20%). Let $\mathbb{F}_q$ be a finite field of q elements. Let $G = \mathrm{GL}_2(\mathbb{F}_q)$ be the group of invertible 2×2 matrices with entries in $\mathbb{F}_q$. Compute the number of conjugacy classes of G and find an element in each conjugacy class.

Problem 3. Let $K \subset \mathbb{C}$ be a quadratic field extension of $\mathbb{Q}$.

(1) (10%) Show that there exists $D \in \mathbb{Q}$ such that $K = \{a + b\sqrt{D} \mid a, b \in \mathbb{Q}\}$ as a subset of $\mathbb{C}$.

(2) (10%) Let $A = \{x \in K \mid x^2 + ax + b = 0 \text{ for some } a, b \in \mathbb{Z}\}$. Show that A is a free $\mathbb{Z}$-module and find a basis.

Problem 4.

(1) (10%) Let p be a prime number and G a subgroup of the symmetric group S_p. Suppose G is transitive and contains a transposition. Show that $G = S_p$.

(2) (10%) Show that the polynomial $x^5 - 4x + 2$ is irreducible and determine its Galois group over $\mathbb{Q}$.

Problem 5 (20%). Let A be a principal ideal domain. Let M be a free module of finite rank over A and $N \subset M$ a submodule. Show that N is free and $\operatorname{rank} N \leq \operatorname{rank} M$.