臺灣大學數學系
105 學年度上學期博士班資格考試題
科目：代數

2016.09.23

QUALIFYING EXAM: ALGEBRA, SEPTEMBER 2016

Problem 1 (15 pts). Let

$$A = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}.$$

Let $B \in \mathrm{M}_{3\times 2}(\mathbb{R})$ and $C \in \mathrm{M}_{2\times 3}(\mathbb{R})$ such that $BC = A$. Find CB.

Problem 2 (15 pts). Let G be a non-trivial finite group. Let p be the smallest prime factor of the order of G. If H is a subgroup of G with index p, show that H is normal in G.

Problem 3 (20 pts). Let G be a finite group of order 455. Show that any 13-Sylow subgroup is in the center of G.

Problem 4 (15 pts). For any positive integer n and a group G, define

$$G[n] = \{x \in G \mid x^n = 1\}.$$

Let G_1 and G_2 be two finite abelian groups. Suppose that for every positive integer n, the cardinalities of $G_1[n]$ and $G_2[n]$ are the same. Show that G_1 and G_2 are isomorphic.

Problem 5 (15 pts). Let A be a Noetherian commutative ring and let B be a finitely generated A-algebra. Let G be a finite group acting on B as A-linear automorphism and let B^G be the set of elements in B fixed by G. Then B^G is an A-algebra. Show that B^G is finitely generated A-algebra.

Problem 6 (20 pts).

(1) Let p be a prime. Show that the symmetric group S_p is generated by any 2-cyle and any p-cycle.

(2) Let E be the splitting field of the polynomial $x^5 - x - 1$ over $\mathbb{Q}$. Show that the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to S_5.