臺灣大學數學系

102 學年度上學期博士班資格考試題

科目：代數 2013.09.27

Problem 1 (15 pts). Prove that a finite group of order 224 is NOT simple.

Problem 2 (15 pts). Let $\mathbb{Q}[x]$ denote the space of polynomials with rational coefficients. Let $f(x) = x^4 + x^3 - px^2 + p$, where p is a prime number. Find all idempotents in the quotient ring $\mathbb{Q}[x]/(f(x))$. In other words, determine all $u \pmod{f(x)} \in \mathbb{Q}[x]/(f(x))$ such that $u^2 = 1 \pmod{f(x)}$.

Problem 3 (20 pts). Let $\mathbb{Z}$ denote the set of integers.

(1) Show that $\mathbb{Z}[\sqrt{-2}]$ is a principal ideal domain.

(2) Find all integers $(a, b) \in \mathbb{Z}^2$ with $b^2 = a^3 - 2$.

Problem 4 (20 pts). Let n be a positive integer and let $M_n(\mathbb{C})$ be the set of n by n matrices with entries in $\mathbb{C}$. Denote by $tr : M_n(\mathbb{C}) \to \mathbb{C}$ the trace function. Let $A, B \in M_n(\mathbb{C})$.

(1) If $tr(A^k) = 0$ for $k = 0, 1, 2, \ldots, n$, prove that $A^n = 0$.

(2) Use (1) to show that if $A(AB - BA) = (AB - BA)A$, then $(AB - BA)^n = 0$.

Problem 5 (15 pts). Let K be the splitting field of the polynomial $x^5 - 6x + 3$ over $\mathbb{Q}$. Show that $\mathrm{Gal}(K/\mathbb{Q}) \simeq S_5$.

Problem 6 (15 pts). Let E be a field and let σ be an automorphism of E. Suppose that $\sigma^4 = 1$ and

$$\sigma(\alpha) + \sigma^3(\alpha) = \alpha + \sigma^2(\alpha) \text{ for all } \alpha \in E.$$

Prove that $\sigma^2 = 1$.