臺灣大學數學系
100 學年度上學期博士班資格考試題
科目：代數　　　　2011.09.16

Notations:
$\mathbb{Z}$: the ring of integers.
$\mathbb{Q}$: the field of rational numbers.
$\mathbb{F}_n$: the finite field with n elements.
$GL_n(F)$: the group of non-singular $n \times n$ matrices over the field F.
$SL_n(F) := \{A \in GL_n(F) \mid \det(A) = 1\}$.
$Z(G)$: the center of the group G.

(1) (20%) Determine the order of $SL_2(\mathbb{F}_3)$, exhibit all Sylow p-subgroups of $SL_2(\mathbb{F}_3)$ with p being a prime dividing $|SL_2(\mathbb{F}_3)|$ and describe the group structure of $SL_2(\mathbb{F}_3)/Z(SL_2(\mathbb{F}_3))$. (Justify your answers)

(2) (20%) Let F be a field and $I = \langle -x^3 + y, x^2y - y^2 \rangle$ be the ideal in the polynomial ring $F[x, y]$. Determine whether the polynomials $f = x^6 - x^5y$ and $g = x^5y - xy^5$ are elements of I. (Justify your answers)

(3) (20%) Let R be a ring. Show that any R-module can be embedded in an injective R-module.

(4) (20%) Find the Galois groups of the polynomials $x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$ and $x^5 - x - 1$ over $\mathbb{Q}$ respectively. (Justify your answers)

(5) (20%) Let A_d be the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{d})$ with $d (\neq 1)$ being square free. Show that A_d is a Euclidean domain for $d = -2, -3, 3, 5$.
($A_d := \{\alpha \in \mathbb{Q}(\sqrt{d}) \mid \alpha \text{ is integral over } \mathbb{Z}\}$)