國立臺灣大學數學系
九十七學年度博士班入學考試試題
科目：統計

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1. Suppose that $X_1, \cdots, X_n$ are iid uniform(0,1) random variables and let $S_n = \sum_{i=1}^n X_i$. Define
$$N = \min\{k : S_k > 1\}.$$
   (a) (5) Find the cdf of S_k.
   (b) (6) Show that $P(N = n) = P(S_{n-1} < 1) - P(S_n < 1)$ and that $E(N) = e$.
   (c) (5) Using the result in (b), write down an algorithm to calculate the value of e by simulation.
   (d) (6) How large should the sample size in (c) be so that you are 95% confident that you have the first four digits of e correct?

2. Suppose $X_1, \cdots, X_n$ is a random sample from a Inverse Gaussian pdf
$$f(x;\mu,\lambda) = \Big(\frac{\lambda}{2\pi x^3}\Big)^{1/2} \exp\Big\{ -\lambda(x-\mu)^2/(2\mu^2 x)\Big\},\ 0 < x < \infty,$$
where $\lambda > 0$ and $\mu \in R$. Let
$$\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i, \quad T = \frac{n}{\sum_{i=1}^n X_i^{-1} - \bar{X}^{-1}}.$$
   (a) (6) Show that $(\bar{X}, T)$ is complete and sufficient for (μ, λ).
   (b) (6) Show that $\bar{X}$ and T are the MLEs of μ and λ.
   (c) (6) Show that $\bar{X}$ and T are independent.
   (d) (8) For $n = 2$, find the distributions of $\bar{X}$ and $n\lambda/T$.

3. Let $X_1, \cdots, X_n$ be a random sample from a Normal(μ, σ^2) distribution, where $\mu \in R$ and $\sigma^2 > 0$ is unknown.
   (a) (6) Find the Cramér-Rao lower bound for the variance of any unbiased estimator of σ^2.
   (b) (6) Suppose μ is known. Can the lower bounded in (a) be attained or not? Explain why.
   (c) (6) Suppose μ is unknown. Can the lower bounded in (a) be attained or not? Explain why.

4. Let X be an observation from the logistic location pdf

$$f(x|\theta) = \frac{\exp(x-\theta)}{\{1+\exp(x-\theta)\}^2}, \quad -\infty < x < \infty, \ -\infty < \theta < \infty.$$

(a) (6) Find the most powerful size α test for testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$.

(b) (6) Show that the test in (a) is UMP size α for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$.

(c) (6) Construct the UMA lower confidence bound for θ with confidence level 1-α.

5. Let $X_1, \cdots, X_n$ be a random sample from a Normal(μ, σ^2) distribution.

(a) (8) If μ is unknown and σ^2 is known, find a score statistic for testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$.

(b) (8) If μ is known and σ^2 is unknown, find a score statistic for testing $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma \neq \sigma_0$.