國立臺灣大學數學系
九十六學年度博士班入學考試試題
科目：統計

2007.05.04

1. (8%) (12%) Let $X_1, \cdots, X_n$ be a random sample from a population with mean μ and variance $\sigma^2 < \infty$. Suppose that $h(x)$ has a second derivative $h^{(2)}(x)$ continuous at μ and $h^{(1)}(\mu) = 0$. Show that $\sqrt{n}(h(\bar{X}) - h(\mu)) \xrightarrow{p} \cancel{0}$ while $n(h(\bar{X}) - h(\mu)) \xrightarrow{d} \frac{1}{2}h^{(2)}(\mu)\sigma^2\chi_1^2$, where $\bar{X}$ is the sample mean of $X_1, \cdots, X_n$.

2. (5%) (5%) (10%) Let $X_1, \cdots, X_n$ be a random sample from the uniform distribution $U(\alpha - \beta, \alpha + \beta)$ where α and β are unknown parameters. Find the uniformly minimum variance unbiased estimators of α, β, and $\frac{\alpha}{\beta}$.

3. Let $X_1, \cdots, X_n$ be a random sample from a population with probability density function

$$f(x|\theta, \nu) = \frac{\theta\nu^\theta}{x^{\theta+1}} 1_{[\nu,\infty)}(x),$$

where θ and ν are unknown positive parameters.

(3a) (5%) (5%) Find the maximum likelihood estimators of θ and ν.

(3b) (10%) Show that the likelihood ratio test of the null hypothesis $H_0 : \theta = 1$ versus the alternative hypothesis $H_A : \theta \neq 1$ has critical region of the form $\{(X_1, \cdots, X_n) : T(X_1, \cdots, X_n) \leq c_1 \text{ or } T(X_1, \cdots, X_n) \geq c_2\}$, where $0 < c_1 < c_2$ and $T(X_1, \cdots, X_n) = \ln\left(\frac{\prod_{i=1}^n X_i}{(min\{X_1, \cdots, X_n\})^n}\right)$.

4. (8%) (12%) Consider the time series $X_1, \cdots, X_n$ which follows a first order autoregressive model so that $(X_t - \mu) = \phi(X_{t-1} - \mu) + \varepsilon_t$, $t = 1, \cdots, n$, where $\mu = E[X_t]$ and ϕ is an unknown parameter, and ε_t's $\overset{i.i.d.}{\sim} N(0, \sigma^2)$. Derive the conditional mean $E[X_t|X_1, \cdots, X_{t-1}]$ and marginal distribution of X_t.

5. Let $X_1, \cdots, X_n$ be a random sample from a population with the probability density function

$$f(x|\lambda) = \lambda \exp(-\lambda x) 1_{(0,\infty)}(x),$$

where λ is a positive parameter.

(5a) (10%) Find a uniformly most powerful size α, $0 < \alpha < 1$ test for the null hypothesis $H_0 : \lambda = \lambda_0$ versus the alternative hypothesis $H_A : \lambda > \lambda_0$, where λ_0 is a known constant.

(5b) (10%) Find a uniformly most accurate $(1 - \alpha)$ confidence interval for λ.