臺灣大學數學系

九十五學年度博士班入學考試題

統計

Jun, 2006

1. (25) Let $X_1, \cdots, X_n$ be a random sample from a continuous distribution with c.d.f. F and p.d.f. f. An estimator of $f(x)$ is

$$\hat{f}(x) = \frac{F_n(x+\delta_n) - F_n(x-\delta_n)}{2\delta_n},\ x \in R,$$

where F_n is the empirical c.d.f. based on the sample and $\delta_n > 0$.

   (a) Find the distribution, bias and variance of $\hat{f}(x)$.

   (b) Find a sequence a_n, constants m and b and conditions on δ_n such that $a_n\{\hat{f}(x) - f(x)\} \to N(m, b)$ in distribution, as $n \to \infty$.

2. (25) Let $X_1, \cdots, X_n$ be a random sample from a Cauchy(m, b) distribution with p.d.f

$$f(x; m, b) = \frac{1}{b\pi}\Big\{1 + \Big(\frac{x-m}{b}\Big)^2\Big\}^{-1},\ -\infty < x < \infty.$$

   (a) Find the distribution of $\bar{X} = \sum_{i=1}^n X_i/n$.

   (b) Find the quartiles of the Cauchy(m, b) distribution.

   (c) Find consistent estimators of m and b.

3. (25) Let $X_1, \cdots, X_n$ be a random sample from a Bernoulli(p) distribution, where $0 < p < 1$. Consider the squared loss.

   (a) Find the Bayes estimator of p when p has a Beta(α, β) prior, which has p.d.f.

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1},\ 0 < p < 1,$$

   where $\alpha > 0$, $\beta > 0$.

   (b) Find the unique minimax estimator of p.

   (c) Compute the relative risk of the minimax estimator of p with respect to the UMVUE $\bar{X}$ and find the range of p where the relative risk is greater than 1.

4. (25) Let X be an observation from $N(\theta-1, 1)$ if $\theta < 0$, $N(0, 1)$ if $\theta = 0$, and $N(\theta+1, 1)$ if $\theta > 0$.

   (a) Find the maximum likelihood estimator of θ.

(b) Find the level-α UMP test for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ for a given θ_0.

(c) Construct the UMA lower confidence bound for θ with confidence level 1-α.