臺灣大學數學系

九十二學年度博士班入學考試題

統計

1. (20 pts.) Let $U_1, U_2, \cdots$ be independent $Uniform(0,1)$ random variables. Let random variable X have the distribution

$$P(X = x) = \frac{1}{(e-1)x!}, \; x = 1, 2, 3, \cdots$$

   Find the distribution of $Z = \min\{U_1, \cdots, U_X\}$.

2. (30 pts.) Let $X_1, \cdots, X_n$, $n \geq 2$ be independent and identically distributed with density

$$f(x, \theta) = \frac{1}{\sigma} \exp\{-(x-\mu)/\sigma\} I_{\{x \geq \mu\}} \, ,$$

   where $\theta = (\mu, \sigma^2)$, $-\infty < \mu < \infty$, $\sigma^2 > 0$.

   1. Find the maximum likelihood estimates of μ and σ and denote them as $\hat{\mu}$ and $\hat{\sigma}$. Are they consistent? Why?
   2. Find sequences of numbers $\{a_n\}$ and $\{b_n\}$ such that

$$\frac{a_n(\hat{\mu} - \mu)}{b_n} \xrightarrow{d} Z,$$

      where Z is a random variable. Specify the distribution of Z.
   3. Find the maximum likelihood estimate of $P_\theta(X_1 \geq t)$ for $t > \mu$.

3. (20 pts.) Consider the model $Y_i = \mu + e_i$, $i = 1, \cdots, n$, where $e_i = (\varepsilon_{i-1} + \varepsilon_i + \varepsilon_{i+1})$ and $\varepsilon_0, \cdots, \varepsilon_n$ are i.i.d. with mean zero and variance σ^2. Find the optimal MSPE predictor of Y_{i+1} based on the past $Y_1, \cdots, Y_i$, which minimizes

$$E(Y_{i+1} - g(Y_1, \cdots, Y_i))^2.$$

   among all functions of $Y_1, \cdots, Y_i$.

4. (15 pts.) Let $X_1, \cdots, X_n$ be i.i.d. inverse Gaussian with parameters μ and λ, where μ is known. That is, each X_i has density

$$\Big(\frac{\lambda}{2\pi x^3}\Big)^{1/2} \exp\Big\{-\frac{\lambda x}{2\mu^2} + \frac{\lambda}{\mu} - \frac{\lambda}{2x}\Big\};\ x > 0,\ \mu > 0,\ \lambda > 0\,.$$

Find the Uniformly Most Powerful test for $H_0 : \lambda \geq \lambda_0$ versus $H_1 : \lambda < \lambda_0$.

5. (15 pts.) Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$ be independent exponential samples. Each X_i has density $f(x|\theta) = \theta^{-1} e^{-x/\theta} I_{(0,\infty)}(x)$ and each Y_j has density $g(y|\lambda) = \lambda^{-1} e^{-y/\lambda} I_{(0,\infty)}(y)$, where $\theta > 0$ and $\lambda > 0$. Find a confidence interval for $\Delta = \theta/\lambda$ with confidence level $1 - \alpha$.