臺灣大學數學系

八十九學年度博士班入學考試題

機率與統計

In this test, there are five questions in total. You must answer the first three questions. For questions 4 and 5, you can pick either one to solve.

1.

(30 points) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with distribution $N(\theta, 1)$ where θ is a real valued parameter. Let

$p = P(X - 1 > a) = \Phi(\theta - a)$.

(11)

Derive the maximum likelihood estimator of p.

(12)

Let $\delta_{1n} = \Phi(\bar{X}_n - a)$ where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$. Determine the limitdistribution of $\sqrt{n}(\delta_{1n} - p)$.

(13)

Let $\delta_{2n} = n^{-1} \sum_{i=1}^{n} 1(X_i > a)$. Here $1(A)$ is an indicator function which is equal to 1 if A occurs. Otherwise, it is equal to 0. Determine the limit distribution of $\sqrt{n}(\delta_{2n} - p)$.

2.

(10 points) The random variable Y has a binomial distribution with an unknown number θ of trials, and known probability of success, 1/2. (Namely, $Y \sim \mathrm{Bin}(\theta, 1/2)$.) Find an approximated 95% confidence interval of θ of the form $[0, c]$ based on Y where c is to be determined.

3.

(30 points) Suppose that the independent pairs of random variables $(Y_1, Z_1), \ldots, (Y_n, Z_n)$ are being observed where Y_j and Z_j are independent, $Y_j \sim N(\xi_j, \sigma^2)$, and $Z_j \sim N(\beta\xi_j, \tau^2)$. Somone claims that Z_j is related to Y_j linearly. Then propose to use the minimizer b of $\sum_{j=1}^{n} (Z_j - bY_j)^2$ as an estimator of β. Derive this estimator b and check whether this estimator is a consistent estimator of β.

4.

(30 points) Let X have a binomial distribution Bin (n, θ) where n is the total number of trials and θ is the probability of success.

(41)

Show that the likelihood-ratio statistics for $H_0 : \theta = 1/2$ versus $H_a : \theta \neq 0$ is equivalent to $|2X - n|$.

(42)

When you only have a normal table, suggest a method for doing a level 0.05 test. (You can assume (41) holds to proceed your proof.)

(43)

Let Y have a binomial distribution Bin $(n, X/n)$. Derive the asymptotic distribution of $\sqrt{n}(Y - X/n)$.

5.

(30 points) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with distribution $N(\xi, 1)$ with $\xi > 0$.

(51)

Show that the MLE is $\bar{X}$ when $\bar{X} > 0$ and does not exist when $\bar{X} \leq 0$. Here $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.

(52)

The MLE exists with probability tending to 1 as $n \to \infty$.

(53)

Show that the estimator

$$\hat{\xi}_n = \bar{x} \text{ when } \bar{X} > 0; \quad \hat{\xi}_n = 1 \text{ when } \bar{X} \leq 0$$

is consistent.