臺灣大學數學系

八十七學年度博士班入學考試題

機率與統計

- 如您決定攻讀機率, 機率考題成績佔成績之70%, 機率部份必須作答四題, 統計部份選答兩題(請註明選取題目); 如您決定攻讀統計, 統計考題成績佔成績之70%, 統計部份必須作答四題, 機率部份選答兩題(請註明選取題目).

機率考題：

以下五題每題20分, 以 $(\Omega, \mathcal{F}, \mathcal{P})$ 代表一機率空間.

(1)

$E_n \in \mathcal{F}$, $n = 1, 2, \cdots$ $\quad \lim_{n\to\infty} \mathcal{P}(E_n) = 0$

(a)

描述一個使 zer-one law 成立的條件.

(b)

若已知 $\sum_n \mathcal{P}(E_n E_{n+1}^c) < \infty$, 證: $\mathcal{P}(\overline{\lim_{n\to\infty}} E_n) = 0$ (A^c 表 A 的 complement).

(c)

若已知 $\sum_n \mathcal{P}(E_n) = \infty$, zero-one law 成立, 且有一常數

$c : \mathcal{P}(E_n \cap E_k) \le c\mathcal{P}(E_n)\mathcal{P}(E_k)$, $\forall\ n \ne k$, 證: $\mathcal{P}(\overline{\lim_{n\to\infty}} E_n) = 1$

(2)

令 X, Y 爲定義於 Ω 之兩個實值隨機變數, 令 μ_1, μ_2 表其機率分佈, 若 X, Y 爲獨立, 證明: ∀ Borel $B \subset R$, $\mathcal{P}(X + Y \in B \,|\, X) = \mu(B - X)$. 將此結果推至 n 個隨機變數之和的情況.

(3)

令 $z(x)$, $x \in R$, 爲有界函數, 且 $F(x)$, $x \in R$, 爲分佈函數. 設 $z(x) = F(x) = 0$, $\forall$ $x \le 0$, 討論函數方程 $Z(x) = z(x) + F * Z(x)$ 之解的存在與唯一 (確實寫下其解), 以機率觀點說明解的含義. (* 表 convolution)

(4)

令 $x(t)$, $t \ge 0$, 表一實值 Markov 過程, 令 $p(s, x, t, dy)$ 表其遷移機率 (transition probability). 何謂過程爲時間齊性 (temporal homogeneous)? 何謂過程爲空間齊性 (spatial homogeneous)? 以 $p\ (s, x, t, dy)$ 描述之, 且予證明.

(5)

就你所瞭解的, 描述數值 R^d 之獨立增分 (independent increments) 隨機過程的 Levy-Ito 理論.

統計考題：

(1)

X and Y are two random variables.

(a)

Find a function of $X, g(X)$, such that

$$\lim_{g} E(Y-g(X))^2.$$

(b)

Find a function of $X, g(X)$, such that

$$\lim_{g} E|Y-g(X)|.$$

Note: You can assume that the minimizers of $E(Y-c)^2$ and $E|Y-C|$ are EY and the median of Y, respectively.

(2)

Let $X_{ij}(i=1,\cdots,p; j=1,\cdots,k)$ be independent $N(\mu_i,\sigma^2)$ variables.

(a)

Find the M. L. E.'s of μ_i, $1 \le i \le p$, and σ^2.

(b)

Suppose K is fixed but $p \longrightarrow \infty$. Show that $\hat{\sigma}^2$ (obtained in (a)) is not a consistent estimate of σ^2.

(3)

Let $X_1,\cdots,X_n$ be a sample from a $N(\mu,1)$ population. Find the U. M. V. U. estimate of $P_\mu[X_1 \ge 0] = \Phi(\mu)$.

(4)

Suppose that the observations are directions around a circle, i.e. of angles between 0 and 2π. Consider the simple null hypothesis that the random variables $Y_1,\cdots,Y_n$ are i.i.d. in the uniform density over $(0,2\pi)$, that is, that the angles are completely random. Suppose that we wish to test for symmetrical clustering around the zero direction.

(a)

Suggest a test statistic when clustering may be around 0 and π.

(b)

Propose a normal approximation to the distribution to the distribution of proposed test statistic under the null hypothesis.