臺灣大學數學系

八十六學年度博士班入學考試題

機率與統計

(一)

機率測度的基本性質

令 $(\Omega, \mathcal{F}, \mathcal{P})$ 為機率空間, $A_1, A_2, \cdots$ 為一列事件 (即 $A_n \in \mathcal{F}$), 證明

(i)

若 $A_1 \subset A_2 \subset \cdots$ 則 $\lim_{n\to\infty} \mathcal{P}(A_n) = \mathcal{P}(\bigcup_{n=1}^{\infty} A_n)$.

(ii)

若 $A_1 \supset A_2 \supset \cdots$ 則 $\lim_{n\to\infty} \mathcal{P}(A_n) = \mathcal{P}(\bigcap_{n=1}^{\infty} A_n)$.

(iii)

(Borel-Cantelli 引理): 若 $(\bigcup_{n=1}^{\infty} A_n) < \infty$, 則 $\mathcal{P}(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k) = 0$.

(二)

獨立隨機率序列的大數法則

令 $X_1, X_2, \cdots$ 為定義於 $(\Omega, \mathcal{F}, \mathcal{P})$ 上的實值獨立隨機變數, 且有共同的機率分佈 (即所謂 IID 序列). 適當的應用 Chebyshev 不等之證明:

(i)

若 X_i 二階級矩 (2nd moment), 則 $\frac{S_n}{n} \longrightarrow \mu$ in probability.

(ii)

若 X_i 有四階級矩 (4th moment), 則 $\frac{S_n}{n} \longrightarrow \mu$ a.s. 於上述

$$S_n = X_1 + \cdots + X_n \quad \mu = EX_i$$

(三)

獨立隨機序列的弱收斂

$X_1, X_2 \cdots$ 如同 (二) 所予者. 令 X_∞ 為一實值隨機變數 (亦定義於 $(\Omega, \mathcal{F}, \mathcal{P})$), 稱:

$\{X_n\}$ 弱 (weak) 收斂於 X_∞ (記成 $X_n \xrightarrow{w} X_\infty$), 若 $Ef(X_n) \longrightarrow Ef(X)$, 對所有 $f : R \longrightarrow R$ 有界連續函數皆成立.

(i)

證明 $X_n \longrightarrow X_\infty$ in probability $\Longrightarrow$ $X_n \xrightarrow{w} X_\infty$

(ii)

反之, 若 $X_\infty = C$ (常數), 則可用 $X_n \xrightarrow{w} C \Longrightarrow X_n \longrightarrow X_\infty$ in probability.

(iii)

令 $\varphi(t) = Ee^{it}$, $t \in R$ (的特徵數函數), 於 X_i 是 Bernoulli 分佈的情況 (即 X_i 取值 $1, 0$, 各爲機率 p, q $(q = 1 - p)$, $0 < p < 1$. 計算 $\frac{S_n}{n^\alpha}$, $0 < \alpha \leq 1$, 之 φ.

(iv)

如何由 (iii) ``看出'', $\frac{S_n}{\sqrt{n}}$ 有中央極限定理.

統計

Let $X_1, \cdots, X_k$ be independent, X_i- $Bin(n_i, P_i)$, $i = 1, \cdots, k$. Here Bin denotes the binomial distribution.

(a)

Deduce that the likelihood ratio statistic for $H_0 : p_1 = \cdots = p_k$ verus $H_a : p_i \notin p_j$ for

$$2 \log \lambda = 2 \sum_{i=1}^{k} \{X_i[\log \widehat{p_i} - \log \widehat{p}] + (n_i - X_i)[\log(1 - \widehat{p_i}) - \log(1 - \widehat{p})]\} .$$

where $\widehat{p_i} = X_i / n_i$, $\widehat{p} = \sum_i x_i / \sum_i n_i$, and λ is the maximum of likelihood ratio and 1.

(b)

Deduce the asymptotic distribution of $2 \log \lambda$ when $n_1, \cdots, n_k \longrightarrow \infty$. Don't use the general theorem which states that it is certain distribution. But you can use the law of large numbers and the central limit theorem.

Remark: You can do (b) without knowing how to do (a).