國立臺灣大學數學系

九十七學年度博士班入學考試試題

科目：機率

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EACH PROBLEM HAS 20 POINTS

In the followings, all r.v.'s (random variables) are defined on some probability space.

(1). 1. If r.v. $Y \geq 0$ and $p > 0$, prove: $E(Y^p) = \int_0^\infty p y^{p-1} P(Y > y) dy$. 2. Discuss the finiteness of the absolute moment of symmetric α-stable law, in cases $\alpha = 2$, $1 < \alpha < 2$, and $0 < \alpha \leq 1$.

(2). 1. Let X_n, $n = 1, 2, 3, \cdots$, be a sequence of i.i.d. r.v.'s. (1) prove that $A := \{\omega : \sum_{i=1}^n X_i(\omega)/n$ converges to a finite number$\}$ is either of $P(A) = 0$, or of $P(A) = 1$. 2. Let the i.i.d. sequence U_n be $Unif[0,1]$ distributed, and $f(x)$ be a continuous function on $[0,1]$; how about 1. in the case for $X_n = f(U_n)$, identify the limit if it has.

(3). Assume that X_n converges to X in distribution, and so does for Y_n to Y. (1) Is it true that $X_n + Y_n$ converges to $X + Y$ in distribution ? prove, or give a counter-example if it is not true. (2) Prove it must be true if the two sequences X_n and Y_n are assumed to be independent.

(4). Let $(X_n, \mathfrak{F}_n)$ be a martingale, and each X_n be in $L^2(dP)$. the difference is $\xi_{m,n} := X_n - X_m, m < n$. Prove 1. $E[\xi_{m,n}^2 | \mathfrak{F}_m] = E[X_n^2 | \mathfrak{F}_m] - X_m^2$. 2. if $\sum_n E\xi_{n-1,n}^2 < \infty$, then the martingale convergence holds both a.s. and in mean square.

(5) X_n be a finite-states irreducible aperiodic Markov chain. 1. prove that there is at most one stationary distribution. You may use " basic limit theorem", but you need to explain how to use it. 2. what means X_n to be doubly stochastic ? In case it is, write down the stationary distribution in 1.