國立臺灣大學數學系
九十六學年度博士班入學考試試題
科目：機率論

2007.05.04

All random variables considered here are given on a probability space $(\Omega, \mathcal{F}, P)$.

1. (25 pts) Let $X_1, X_2, \ldots$ be i.i.d. with $E(X_1) = 0$ and $E|X_1|^p < \infty$ where $1 < p < 2$. If $S_n = X_1 + X_2 + \cdots + X_n$, then $S_n/n^{1/p} \to 0$ a.s.

2. (20 pts) Let $X_1, X_2, \ldots$ be i.i.d. with $X_i \geq 0$, $E(X_i) = 1$, and $var(X_i) = \sigma^2 \in (0, \infty)$. Determine the distribution of $\sqrt{S_n} - \sqrt{n}$ as $n \to \infty$ where $S_n = X_1 + X_2 + \cdots + X_n$.

3. (30 pts) Suppose we start out at time 2 with one black ball and one white ball in an urn. Then at each time we draw a ball at random from the urn, and replace it together with a new ball of the same color. Let X_n denote the number of white balls at time n. Write $M_n = X_n/n$ which is the fraction of white balls at time n.

   (a) Determine $E(M_{n+1}|X_{2,n})$. Here $X_{2,n}$ denote the portion $X_2, X_3, \ldots, X_n$ of the process from time 2 up to time n.

   $$\begin{aligned} E\left(\frac{X_{n+1}}{n+1}|X_{2,n}\right) &= \frac{1}{n+1}\left[(X_n+1)\frac{X_n}{n} + X_n\left(1 - \frac{X_n}{n}\right)\right] \\ &= X_n/n = M_n. \end{aligned}$$

   (b) Show that as the time $n \to \infty$, the fraction M_n approaches a limit with probability 1.

   (c) What is the distribution of this limiting fraction?

4. (25 pts) Let $Y_1, Y_2, \ldots$ be nonnegative i.i.d. random variables with $EY_m = 1$ and $P(Y_m = 1) < 1$.

   (a) Show that $X_n = \prod_{m \leq n} Y_m$ defines a martingale.

   (b) Show $X_n \to 0$ a.s.

   (c) Show that $(1/n) \log X_n \to c < 0$.