臺灣大學數學系

九十五學年度博士班入學考試題

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ENTRANCE EXAM FOR PHD PROGRAM

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All random variables are given on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

1. (30 pts)

   (a) For any random variable U show that
   $$\sum_{n=1}^{\infty} \mathbf{P}(|U| \geq n) \leq \mathbf{E}[\,|U|\,] \leq \sum_{n=0}^{\infty} \mathbf{P}(|U| \geq n).$$
   Note that $\mathbf{E}[\,|U|\,]$ can be infinite.

   (b) For a sequence of independent, identically distributed random variables (V_n), show that $\dfrac{V_n}{n} \to 0$ almost surely (a.s.) as $n \to \infty$ if and only if $\mathbf{E}[\,|V_1|\,] < \infty$.

2. (20 pts)

   (a) Show that if $X_n \to X$ in probability then X_n converges to X weakly.

   (b) Conversely, show that if X_n converges weakly to a constant $c \in \mathbb{R}$ then $X_n \to c$ in probability.

3. (30 pts) Let $Y_n, n \geq 1$, be a sequence of independent random variables with $\mathbf{E}[Y_n] = 0, \mathbf{E}[Y_n^2] < \infty$. Denote $M_n = Y_1 + \cdots + Y_n$ and sub σ-fields $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n), n \geq 1$, and $\mathcal{F}_0 = \{\Omega, \emptyset\}$.

   (a) Prove that M_n is a martingale and M_n^2 is a sub-martingale, both are with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$.

   (b) Find all possible sequence of random variables $A_n \in \mathcal{F}_{n-1}, n \geq 1$, such that $A_{n+1} \geq A_n$, and $M_n^2 - A_n$ is a mean zero $\mathcal{F}_n$-martingale.

4. (20 pts) Let Z be a random variable with $\mathbf{E}[\,|Z|\,] < \infty$, and $\mathcal{G}_n, n \geq 1$, be any increasing sub σ-fields of $\mathcal{F}$.

   (a) Show that $Z_n = \mathbf{E}[\,Z \,|\, \mathcal{G}_n\,], n \geq 1$, is a uniformly integrable $\mathcal{G}_n$-martingale.

   (b) Does Z_n converge in L^1? Does Z_n converge almost surely? What is the limit if Z_n converges?