臺灣大學數學系

九十二學年度博士班入學考試題

機率

1.(20 pts) (a) Let X_n, Y_n be two sequences of r.v.'s defined on a probability space (Ω, F, P). If $\sum_n P\{X_n \neq Y_n\} < \infty$, then prove that $\frac{1}{n}\sum_{j=1}^{n}(X_j - Y_j)$ converges almost surely.

(b) Under the condition of (a), prove that $\frac{1}{n}\sum_{j=1}^{n} X_j$ converges or diverges in the same way as $\frac{1}{n}\sum_{j=1}^{n} Y_j$. pt 2.(20 pts) (a)Let X_n be a sequence of independent and identically distributed r.v.'s which are positive and not identically zero a.s. Regard X_n as lifespan of certain object undergoing a process of renewals. Given a time instant t, let $N(t)$ denote the number of renewals up to and including the time t. Let $S_n = \sum_{j=1}^{n} X_j$. Express the event $\{N(t) = n\}$ in terms of S_n.

(b) Use SLLN of S_n to give the limit $\lim_{t\to\infty}\frac{N(t)}{t}$. pt 3.(20 pts) Let X have the normal distribution $N(0,1)$. Find the distribution function, probability density function and characteristic function of X^2. pt 4.(20 pts) Define a double array of r.v.'s as follows. For each $n \geq 1$ let there be k_n independent r.v.'s $X_{nj}, 1 \leq \jmath \leq k_n$,, where $k_n \to \infty$ as $n \to \infty$. Assume that $EX_{nj} = 0, \sum_{j=1}^{k_n}\sigma^2(X_{nj}) = 1$, and that $\sum_{j=1}^{k_n} E|X_{nj}|^3 \to 0$ as $n \to \infty$. Let $S_n = \sum_{j=1}^{k_n} X_{nj}$, Use characteristic functions to show roughly that S_n converges in distribution to $N(0,1)$. pt 5.(20 pts) Let X_n be a sequence of r.v.'s and F_n be an increasing sequence of sub-sigma algebras of F. Assume that each X_n is measurable with respect to F_n. (a) What is meant by optional r.v.(stopping time) for a positive integer valued r.v. α ? What is meant by the optional(stopped) r.v. X_α ? What is meant by the optional(stopped) sigma algebra F_α generated by α ?

(b) Let α, β be two bounded optional r.v.'s such that $\alpha \leq \beta$. Prove that if X_n, F_n is a martingale, then so is $X_\alpha, X_\beta, F_\alpha, F_\beta$.