國立臺灣大學數學系
九十六學年度博士班入學考試試題
科目：偏微分方程

2007.05.04

There are five problems in this exam (2 pages). Please answer each problem and explain your arguments as much detail as possible.

1. (a)(10%) Let Ω be an open subset in $\mathbb{R}^n$ with smooth boundary. Assume that $a(x) = (a_{ij}(x))_{i,j=1}^n \in L^\infty(\Omega)$ is a positive-definite matrix. Consider the boundary value problem:

$$\begin{cases} \sum_{i,j=1}^n \partial_{x_j}(a_{ij}\partial_{x_i} u) = 0 \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = f, \end{cases} \tag{1}$$

where $f \in H^{1/2}(\partial\Omega)$. Prove that (1) exists one and only one solution $u \in H^1(\Omega)$. (Hint: Lax-Milgram Theorem)

(b)(10%) Let $F : \bar{\Omega} \to \bar{\Omega}$ be a diffeomorphism with $y = F(x)$. Then $\tilde{u}(y) := (u \circ F^{-1})(y)$ is the solution of

$$\begin{cases} \sum_{i,j=1}^n \partial_{y_j}(\tilde{a}_{ij}\partial_{y_i} \tilde{u}) = 0 \quad \text{in} \quad \Omega, \\ \tilde{u}|_{\partial\Omega} = \tilde{f}, \end{cases}$$

where $\tilde{f} = f \circ F^{-1}|_{\partial\Omega}$. Write down explicitly the new matrix $\tilde{a} = (\tilde{a}_{ij})$.

2.(a)(10%) Show that in $\mathbb{R}^3$

$$G(x,y) = \frac{1}{4\pi}\frac{e^{-|x-y|}}{|x-y|}$$

is the fundamental solution of the operator $-\Delta + 1$.

(b)(10%) Without using the explicit form of the fundamental solution, prove that any derivative of the fundamental solution of $-\Delta + k$ in $\mathbb{R}^3$ with $k > 0$ decays faster than any polynomial order as $|x| \to \infty$.

3.(15%) Consider the Schrodinger equation:

$$\frac{1}{i}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{in} \quad (t,x) \in (0,1) \times (0,\infty).$$

Show that if $u(t,0) = 0$ for all $t \in [\alpha, \beta]$ with $0 < \alpha < \beta < 1$, then $u(t,x) \equiv 0$ in $[\alpha, \beta] \times [0, \infty)$.

4. Define the heat equation:

$$u_t - \Delta u = 0, \quad t \in (0, \infty), \ x \in \mathbb{R}^n.$$

(a)(10%) Let $\phi(x)$ be a bounded continuous function in $\mathbb{R}^n$. Show that

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \phi(y)dy$$

satisfies the heat equation and the initial condition

$$\lim_{t \to 0^+} u(x,t) = \phi(x) \quad \text{for any } x \in \mathbb{R}^n.$$

Furthermore, if $\phi \geq 0$ and $\|\phi\|_{L^1(\mathbb{R}^n)} < \infty$ then

$$\|u(\cdot, t)\|_{L^1(\mathbb{R}^n)} = \|\phi\|_{L^1(\mathbb{R}^n)}.$$

(b)(10%) Let u be as in (a). Show that for $p > 0$, the relation

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} = \|\phi\|_{L^p(\mathbb{R}^n)}$$

holds for any pair of ϕ and u only when $p = 1$.

(c)(10%) Let u be as in (a) with $\|\phi\|_{L^1(\mathbb{R}^n)} < \infty$. Show that $\lim_{t \to \infty} u(x,t) = 0$ *uniformly* in $\mathbb{R}^n$.

5.(a)(5%) Consider the wave equation in $(t, x) \in \mathbb{R} \times \mathbb{R}^3$:

$$\partial_t^2 u - c^2 \Delta u = 0.$$

Given an initial point $P = (0, x_0)$. Describe the domains of influence of P.

(b)(10%) Consider the one-dimensional wave equation with boundary condition:

$$\partial_t^2 u - c^2 \partial_x^2 u = 0 \quad \text{in} \quad t > 0, x > 0$$

and

$$u(t, 0) = 0.$$

Given $P = (t_0, x_0)$ with $t_0 > 0$, $x_0 > 0$. Determine the domain of dependence of P.

(Note: in both (a) and (b) c is a positive constant.)