臺灣大學數學系

九十五學年度博士班入學考試題

偏微分方程

Jun, 2006

1. (40 pts) Let $u : \mathbb{R}^n \to \mathbb{R}$ be a nonconstant harmonic function, where $n \geq 2$. Answer the following questions:

   (i) Must the function u be bounded? (10 pts)

   (II) Must the function u be smooth? (10 pts)

   Assume $u(0) = 0$. Define a function N by

$$N(r) = \frac{r \int_{B_r} |\nabla u|^2 dx}{\int_{\partial B_r} u^2 dS}, \quad \forall r > 0,$$

   where $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ is the standard ball with radius r and center at the origin. Besides, $\partial B_r = \{x \in \mathbb{R}^n : |x| = r\}$. Answer the following questions:

   (iii) Must the function N be monotone increasing? (10 pts)

   (iv) Must the limit $\lim_{r \to 0+} N(r)$ exist? (10 pts)

   Prove or disprove all your answers.

2. (30 pts) Choose one of the following equations which may NOT have regularity theorem:

   (A) heat equation

   (B) wave equation

   (C) Burgers' equation without viscosity.

   Prove or disprove your answer.

3. (30 pts) Consider two minimization problems given by

(I) Minimize $E[u] = \int_{\mathbb{R}^2} |\nabla u|^2 + u^2 - u^4$ for $u \in H^1(\mathbb{R}^2)$

(II) Minimize $F[u] = \int_\Omega |\nabla v|^2 + v^4$ for $v \in H_0^1(\Omega)$ and $\int_\Omega v^2 = 1$,

where Ω is a bounded smooth domain in $\mathbb{R}^2$. Can the associated minimizer exist for problem (I) and (II)? Prove or disprove all your answers.