台灣大學數學系

九十三學年度博士班入學考試題

PDE

June 4, 2004

(1)

(a) Please state the Cauchy-Kovalevskaya theorem. (b) Solve the equation

$$x^2 u_x - u u_y = x^3, u(x,0) = x^2.$$

(2)

Let $u(x,t)$ be a solution of the two-dimensional wave equation with initial data vanishing outside a circle, show that $u(x_o, t) = O(t^{-1})$ for fixed x_o and $\sup_x u(x,t) = O(t^{-\frac{1}{2}})$ as $t \to \infty$. (Hint: use the formula

$$u(x_o,t) = \int_{|x-x_o|\le ct} \frac{u_t(x,0)}{2\pi c[c^2t^2 - |x-x_o|^2]^{\frac{1}{2}}}\, dx + \frac{\partial}{\partial t}\int_{|x-x_o|\le ct} \frac{u(x,0)}{2\pi c[c^2t^2 - |x-x_o|^2]^{\frac{1}{2}}}\, dx.$$

(3)

Let $u(x,t)$ satisfy

$$u_t = u_{xx} + f(u), x \in [0,1], t > 0 u(0,t) = u(1,t) = 0$$

(a)

If $f(u) = -u$, show that $u \to 0$ as $t \to \infty$.

(b)

Assume that each solution of the ODE $v_t(t) = f(v(t))$ tends to 0 as $t \to \infty$. Prove that $u \to 0$ as $t \to \infty$.

(4)

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and u be a $C^2(\bar{\Omega})$ solution of

$$\triangle u - u = 0 \quad \text{in } \Omega, u = g \quad \text{on } \partial\Omega.$$

(a)

Show that

$$\int_\Omega [|\nabla u(x)|^2 + u(x)^2]\, dx \le \int_\Omega [|\nabla v(x)|^2 + v(x)^2]\, dx$$

for all $C^2(\bar{\Omega})$ function v with boundary value g.

(b)

Let $I_u(x) = |\nabla u(x)|^2 + u(x)^2$ and $\{x : |x-y| \le r\} \subset \Omega$. Prove that

$$I_u(y) \le \frac{1}{\omega_n r^{n-1}} \int_{|x-y|=r} I_u(x)\, dS_x,$$

where $\omega_n r^{n-1}$ is the surface area of the sphere $|x-y| = r$.