臺灣大學數學系

九十一學年度博士班入學考試題

偏微分方程

There are problems A to D. You have to do all the problems in A , and one of the problem B,C and D.

A.

Solve the following PDE problems explicitly. You can express the solution in closed form, or in series form, or in integral form, etc..

(a)

Find the general solution of $yu_{xx}+(x+y)u_{xy}+xu_{yy}=0$ in the plane domain defined by $x\neq y$.

(b)

Solve the Cauchy problem

$$\{u_t=u_{xx}-xu \quad \text{for} \quad x\in\mathbf{R}, t>0 u(x,0)=f(x),$$

where $f(x)$ is a bounded integrable continuous function on $\mathbf{R}$.

(c)

Fine a global weak solution of

$$xu_t+\frac{1}{2}(u^2)_x-x^2u_x=0 \quad \text{for} \quad 0<x<a, t>0,$$

$$u(x,0)=0 \quad \text{for} \quad 0<x<a, \quad \text{and} \quad u(a,t)=a \quad \text{for} \quad t>0,$$

where a is a positive constant.

B.

Let $U=\{(x,y)| \quad x^2+y^2<1\}, U_1=\{(x,y)|x^2+y^2<1, x>0\}$**, and** $U_2=\{(x,y)|x^2+y^2<1, x<0\}$**. A function** $u\in C(\overline{U})\cap C^2(U_1)\cap C^1(\overline{U_1})\cap C^2(U_2)\cap C^1(\overline{U_2})$ **satisfies**

$$\Delta u+A_1(x,y)u_x+B_1(x,y)u_y=0 \quad \text{in} \, U_1,$$

$$\Delta u+A_2(x,y)u_x+B_2(x,y)u_y=0 \quad \text{in} \, U_2,$$

$$k_1 u_x(0+,y) = k_2 u_x(0-,y),$$

where $A_i(x,y)$ **and** $B_i(x,y)(i=1,2)$ **are continuous bounded functions in** U_i. k_i **are positive constants. Prove that if** $u \geq 0$ **on** $x^2+y^2=1$, **then** $u(x,y) \geq 0$ **in** U.

C.

Let U **be a bounded domain in** $\mathbf{R}^n$ **with** C^1 **boundary.Denote the exterior unit normal to** ∂U **by** ν. **Consider the wave equation**

$$u_{tt} = c^2 \Delta u \quad \text{for} \quad (x,y) \in U \times (0,\infty),$$

$$T\frac{\partial u}{\partial \nu} + \sigma u = 0 \quad \text{on} \quad \partial U \times (0,\infty).$$

where c, T **and** σ **are positive constants.**

(a)

Prove that there exists some costants ρ such that the total energy

$$E(t) = \int_U [\frac{\rho}{2}u_t^2 + \frac{T}{2}|Du|^2]dx + \int_{\partial U} \frac{\sigma}{2}u^2 dS$$

is constant in t.

(b)

Define $E_0(t) = \int_U (x,y)^2 dx$. Prove the energy estimates

$$E_0(t) \leq 2E_0(0) + \frac{4t^2}{\rho}E(0),$$

$$E_0(t) \leq e^t E(0) + \frac{2}{\rho}E(0)(e^t - 1).$$

D.

Let U **be the unit ball in** $\mathbf{R}^n$. P^* **is a point in** ∂U. **Suppose that** $u \in C(\overline{U} - \{P^*\})$ **is harmonic and bounded in** U **, and** $u=0$ **on** $\partial U - \{P^*\}$,

prove that $u=0$ **in** U. **If** u **is unbounded in** U, **must the conclusion be true ?**