臺灣大學數學系

八十九學年度博士班入學考試題

偏微分方程式

There are problems A to F. You have to solve 4 problems. Each has 25 points.

A.

Let $f(t)$ be a C^∞ function in R such that $f(t)=0$ for $t<1$ or $t>2$, and $f(\frac{3}{2})=1$. Show that $xu_y - yu_x = f(x^2+y^2)$ has no weak solution in $R^2-\{(0,0)\}$, i.e. there exists no locally integrable function $u(x,y)$ in $R^2-\{(0,0)\}$ such that, for any C^∞ function $\phi(x,y)$ with compact support in $R^2-\{(0,0)\}$, we have

$$\int_{R^2}[u(y\phi)_x - u(x\phi)_y - f(x^2+y^2)\phi]\,dxdy = 0.$$

B.

The telegraph equation in a finite cable $0 \le x \le l$ is given by

$$\begin{cases} c^2u_{xx} = u_{tt} + 2\beta u_t + \alpha u, & \text{for } 0<x<l,\quad t>0 \\ u(0,t)=0, \quad u(l,t)=0, & \text{for } t \ge 0 \\ u(x,0)=f(x), \quad u_t(x,0)=g(x), & \text{for } 0 \le x \le l \end{cases}$$

where $c>0$. α and β are constants with $\beta^2 \ge \alpha$. l is the length of the cable. $f(x)$ and $g(x)$ are smooth.

(a)

Prove that the solution, if exists, is unique by showing that

$$\frac{dE}{dt} \le 2(1+\beta^2-\alpha)E \quad \text{whenever } f(x)=g(x)\equiv 0$$

where $E(t)$ is the total energy defined by

$$E(t) = \frac{1}{2}\int_0^l [(u_t+\beta u)^2 + c^2u_x^2 + u^2]e^{2\beta t}\,dx.$$

(b)

Use the separation of variables to solve $u(x,t)$ in series form. Show that this solution is indeed a classical solution.

C.

Find those real numbers α so that the problem

$$\Delta u = 0 \text{ in } x^2+y^2 \le 1, \text{ with boundary condition } \frac{\partial u}{\partial n} - \alpha u = 0 \text{ on } x^2+y^2 = 1$$

has a nonzero classical solution $u(x,y)$. Must such solution satisfy the maximum principles? Here n denotes the exterior normal to the boundary $x^2+y^2=1$.

D.

Show that the initial-boundary problem for the heat equation

$$\begin{cases} u_t = u_{xx} & \text{for } 0 < x < \infty, t > 0 \\ u(x,0) = 0 & \text{for } x > 0 \\ u(0,t) = g(t) & \text{for } t \ge 0 \end{cases}$$

has the unique solution

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s)\, ds.$$

Here $g(t)$ is a given smooth bounded function with $g(0) = 0$.

E.

Consider the initial value problem

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0, \quad u(x,0) = h(x), \quad x \in R, \quad t > 0.$$

(a)

Find the solution of this problem, and show that it becomes singular when $a(h(s))$ is a nonincreasing function.

(b)

Find the breaking time (i.e., when the singularity occurs) in the case that $a(u) = u$ and $h(x) = exp(-x^2)$.

F.

Consider the single conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad u(x,0) = u_0(x), \quad x \in R, \quad t > 0,$$

where $f', f'' > 0$, and $u_0 > 0$.

(a)

Prove that the jump condition across a discontinuous curve Γ in the $x - t$ plane is

$$\sigma(u_r - u_l) = f(u_r) - f(u_l),$$

where $\sigma = dx/dt$ is the slope of the tangent line on Γ and u_r, u_l are the solution to the left and right of the discontinuity, respectively.

(b)

Prove that there exist a constant E for which $\partial u/\partial x < E/t$ for all $t > 0$.